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APPLICATIONS OF TENSOR CALCULUS IN POINT MECHANICS, ELASTICITY THEORY, AND FRACTURE MECHANICS

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Abstract. *This paper presents fundamental applications of tensor calculus in point mechanics, elasticity theory, and fracture mechanics. The inertial force is formulated in vector form, and its contravariant and physical coordinates are derived in generalized curvilinear coordinate systems. The stress tensor is presented in vector and tensor forms, together with its physical, covariant, and contravariant components. The Lagrange strain tensor is also discussed in curvilinear orthogonal coordinate systems.*
Special attention is devoted to the kinematics of deformable bodies. The vector of the total reduced transformation of an oriented infinitesimal line element is defined, and the corresponding vector of proper rotation is introduced together with its physical, covariant, and contravariant tensor coordinates.
The paper further outlines several basic concepts of fracture mechanics. The potential energy density, the Eshelby energy tensor, and the J and M integrals are presented in tensor form. Their physical interpretation is discussed, emphasizing the significance of the J integral as a generalized force associated with crack propagation.

Key words: *Tensor calculus, Covariant coordinates, Contravariant coordinates, Material point, Deformable body, Stress tensor, Strain tensor, Rotation angle, Fracture mechanics, **J** integral.*

1. INTRODUCTION

Academician Tetomir P. Anđelić (1903-1993) is the author of a unique monograph (see Reference [1]) on Tensor Calculus, in the Serbian language. It is still a unique reference,

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according to which many generations of postgraduate studies have studied Tensor Calculus. He was, also, a professor of the first scientific course on Tensor Calculus in Europe at the famous Center for Mechanics in Udine, Italy. His lectures were published in Springer's publication (see Ref. [2], as well as Fig. 1). Here we, also, highlight Rational Mechanics [2], as a companion publication of the application of Tensor Calculus in individual chapters of Mechanics of systems of material points, which is still relevant today.

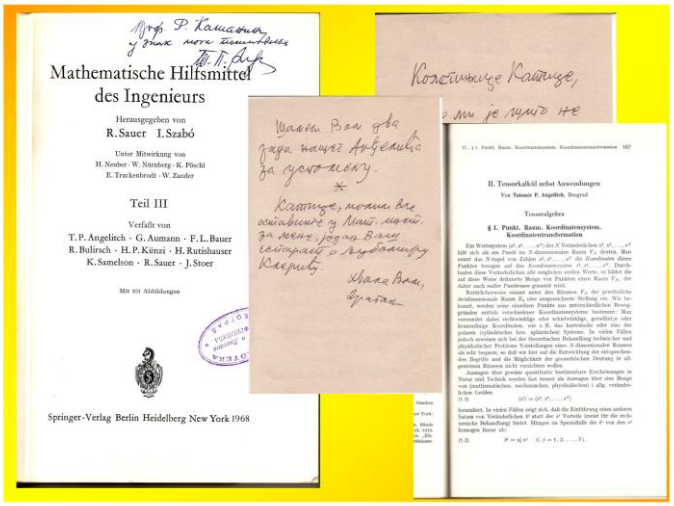


Figure 1. View of the cover and separate pages of the published Ref. [2] of the lectures by Academician Tatomir P. Anđelić, given on Tensor Calculus at the famous Center for Mechanics in Udine, Italy

We emphasize Professor Dr. Eng. and Dipl. Mathematician Danilo P. Rašković (1910-1985) as a pioneer who introduced tensor calculus among students of technical sciences at technical and mechanical engineering faculties in Serbia and Yugoslavia through study programs and excellent textbooks, almost scientific monographs on Kinematics and Theory of Elasticity. (see Refs [4-6], as well as Figs 2 and 3). We emphasize that the Theory of Elasticity, Ref. [5], as an Appendix contains a chapter on Tensor Calculus, adapted for understanding by postgraduate students at technical faculties.

Prof. Dr. Katica (Stevanović) Hedrih (1944-), taught Electrodynamics to mechanical engineering students at the Faculty of Mechanical Engineering in Niš and wrote the textbook Selected chapters of Theory of Elasticity for full-time and postgraduate students, in which she included her original results, which contain individual chapters of the theory of elasticity presented in tensor form, as well as transformations of individual key equations in curvilinear coordinate systems (see Refs [7-9]).

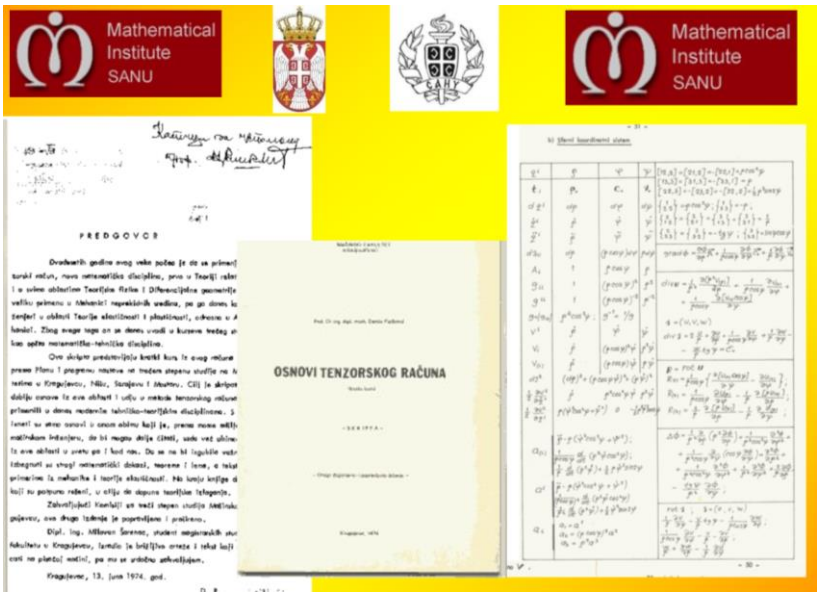


Figure 2. Several characteristic pages from the publication Fundamentals of Tensor Calculus by Damil P. Rašković, Ref. [4]

Professor Akitsugu Kawaguchi, founder of the Tensor Society and founder of the Tensor magazine, was born on April 8, 1902, and academician Tatomir P. Anđelić, a year later in 1903. They were almost peers in age.

Professor Akitsugi Kawaguchi, founder and life president of the Tensor Society and founder of the unique Tensor Journal and the current president of the Tensor Society and Editor-n-Chief of the Tensor Journal is Professor Tomoaki Kawaguchi, see Figure 4.

Academician Tatomir P. Anđelić, and Professor and founder of the Tensor Journal, Akitsugi Kawaguchi, founder of the Tensor society, collaborated scientifically, and in 1970, Professor Akitsugi Kawaguchi was a guest of Academician Tatomir Anđelić and the Serbian Academy of Sciences in Belgrade.

Akitsugu Kawaguchi is the Founder of Tensor society and first president of the Tensor society since 1938. Tensor society is of international nature. In 1970, after his participation to the International Congress of Mathematicians held in Nica (France) on September 1-10, he was invited to the V. Kongres matematičara, fizičara i astronoma Jugoslavije (Fifth Congress of Mathematics, Physics and Astronomy in Yugoslavia) held in Ohrid (Makedonia, Yugoslavia) on September 14-19, as a guest and delivered a one-hour lecture "*On differential Geometry of higher order manifolds*" there.

Afterward, the Serbian Academy of Sciences and Arts in Belgrade invited him to deliver a lecture and then was invited by the University in Novi Sad in order to talk with the staffs. Before this participation to the Congress in Nice, on invitation be visited the Mathematical Institute in Oberwolfach (Schwarcwald, West Germany) in the end of August.

Figure 3. Cover of the publication Fundamentals of Tensor Calculus and Monography Theory of Elasticity by Danilo P. Rašković (1910-1985), References [3, 4]; in the middle, photographs of the author Danilo Rašković and reviewers Academician Tatomir P. Anđelić (1903-1993) and Prof. Dr. Vlatko Brčić (1919 -2000), as well as Prof. Dr. Katica (Stevanović) Hedrih, proofreader and editor of the Preface to the Theory of Elasticity, and accompanying appendices and indexes (Note: the author passed away before the publication of his work- Monograph)

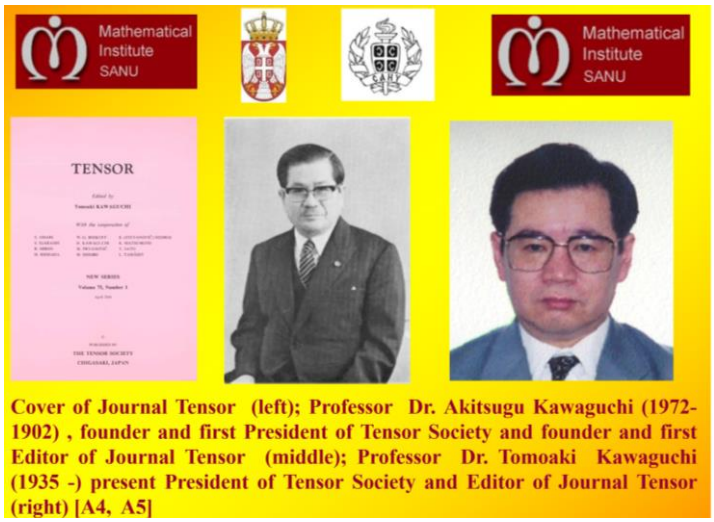


Figure 4. Cover of Tensor Journal and photos of Akitsugu Kawaguchi, founder of Tensor Journal, and Tomoaki Kawaguchi, current Editor-in-Chief of Tensor Journal

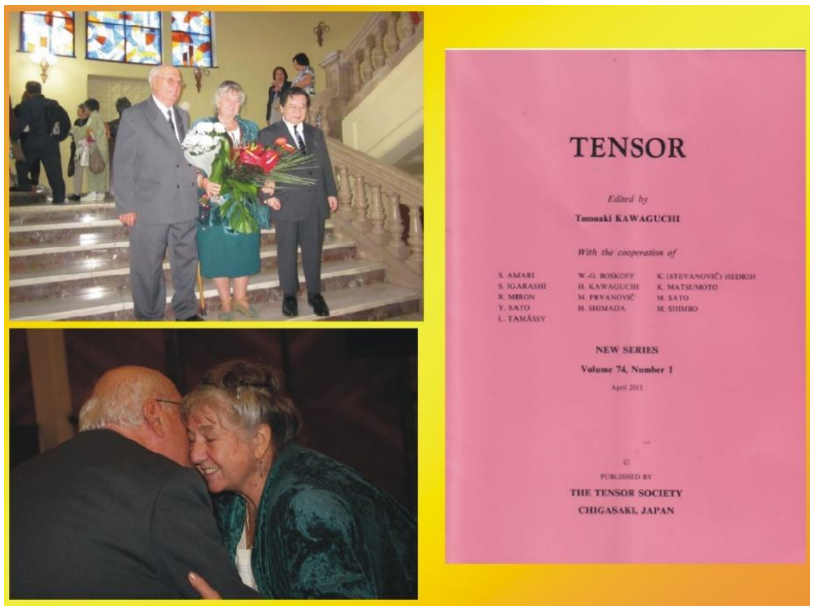


Figure 5. Photos of the President of the Tensor Society Tomoaki Kawaguchi and the Editor-in-Chief of the Tensor Journal and Academician Radu Miron at the University of "Alexander Ion Cusa" in Iași, Romania, when the membership of Professor Katica (Stevanović) Hedrich in the Editorial Board of the Tensor Journal was announced

At the University of Tokyo, a group of mathematicians collaborated on the "Natural philosophy of Kazo Kondo", in which Akitsugi Kawaguchi and Tomoaki Kawaguchi were important members and researchers, in addition to an even larger number of other Japanese mathematicians. Academician Tatomir P. Anđelić was very interested in the scientific results of that group and that theory, in addition to the field of Tensor Calculus.

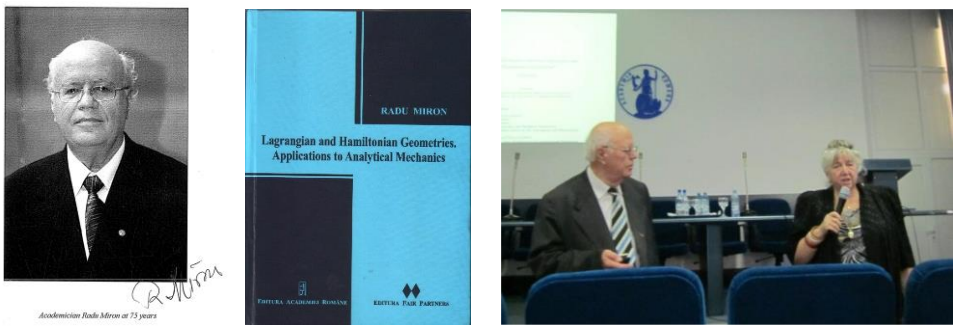
Researchers from the Universities of Belgrade, Novi Sad and Niš collaborated with these Japanese scientists and the Tensor Society and were members of the Tensor Society, among whom author of this paper would like to highlight Academician Božidar Vujanović and Academician Mileva Prvanović. A number of them published their works in the journal Tensor, which is highly scientifically valued and was on the list of MI SANU for evaluating scientific results of projects in Serbia, before the creation of KoBSON. This cooperation, established by Academician Taromir Anđelić, was continued by younger generations, with Professor Tomoaki Kawaguchi and the Tensor Society. An exceptional collaboration was established by Professor Katica (Stevanović) Hedrich, who, at the suggestion of the Editor-in-Chief of the Tensor journal Tomoaki Kawaguchi and the leading member of the Editorial Board, Academician Radu Miron, was appointed a member of the Editorial Board of the Tensor journal, see Figure 5. At that time, Academician Mileva Prvanović, a member of the Editorial Board of the Tensor journal, congratulated her and explained that she herself had been appointed to the Board as the successor to her late mentor. She also emphasized that Katica Hedrih had been elected solely on the basis of her outstanding scientific contributions to tensor applications in mechanics, which were highly appreciated by Tomoaki Kawaguchi and Academician Radu Miron. It was a truly sincere congratulation and a significant assessment of the scientific contribution, coming from an academician of the Serbian Academy of Sciences and Arts, Mileva Prvanović, remembers Prof Katica Hedrih.

President Tomoaki Kawaguchi was an invited plenary speaker at the International Symposium on Nonlinear Sciences held in 2003 in Niš. Academician Mileva Prvanović and Katica (Stevanović) Hedrich participated in a number of scientific conferences organized by the Tensor Society and President Tomoaki Kawaguchi, which were held in Japan, Romania and India. Some of these conferences were attended by some researchers from Serbia, among whom were participants, for example at the Conference in Calcutta, professors Ljubica Velimirović and Zoran Rakić.

The Department of Mechanics of the Faculty of Mechanical Engineering in Niš collaborated with scientists from the University of "Alexander John Cusa" in Iashi who were members of the Japanese Tensor Society. Among them, the most prominent scientist was Academician Radu Miron, a regular member of the Romanian Academy of Sciences. He was a visiting fellow at the Faculty of Mechanical Engineering in Niš in 2007, and at the Mathematical Institute of the Serbian Academy of Sciences and Arts, together with his collaborator Professor Camellia Frigioiu. On November 23, 2007, Academician Radu Miron gave a lecture entitled: "Lagrangian and Finslerian Mechanics of Nonconservative Mechanical Systems" at the Laboratory for Experimental Mechanics of the Faculty of Mechanical Engineering in Niš.

Before that, he was a visiting fellow at the Mathematical Institute of the Serbian Academy of Sciences and Arts. Radu Miron was an active member of the Tensor Society and a member of the Editorial Board of the Tensor journal and a close associate and friend of the President of the Tensor Society, Tomoaki Kawaguchi. Here we highlight his monograph entitled: "Lagrangian and Hamiltonian Geometries – Applications to

Analytical Mechanics", Editura Academiei Romane and Editura Fair Partners, Bucuresti, 2011 (see Reference [49]). In that monograph he also cited the work of scientists from Serbia. In the appendix is the Data Composition in the form of Fig. 6.



a) b) c)

Figure 6. a) Portrait of Academician Radu Miron with signature; b) Cover of the monograph " Lagrangian and Finslerian Mechanics of Nonconservative Mechanical Systems"; C) Academician Radu Miron, lecturer and chair Professor Katica (Stevanović) Hedrich at the Department of the Romanian Academy of Sciences in Iashi, during the conference on differential geometry of the Tensor Society from Japan.

A Special Issue of the International Journal of Non-Linear Mechanics entitled "Elements of Mathematical Phenomenology and Phenomenological Mapping in Nonlinear Dynamics" was published in Volume 73 (2015). The issue comprises papers addressing different thematic aspects of nonlinear dynamics and highlights the importance of the phenomenological approach in the study of physically diverse systems, see Ref. [10].

This paper presents a concise phenomenological overview of nonlinear functional spaces, coordinate systems, and nonlinear coordinate transformations as fundamental elements of mathematical phenomenology and phenomenological mappings.

It is noted that the concepts of Elements of Mathematical Phenomenology and Phenomenological Mappings must start with difference between linear and nonlinear transformations of coordinates and point out fundamental difference between two coupled areas of sciences differential geometry and mechanics (including kinematics and dynamics).

Mathematical Phenomenology and Phenomenological Mappings are captured in three works of Professor Mihailo Petrović (1868–1943), a Serbian academician and mathematician and large scientist, who was one of three doctoral students of world important scientist Jules Henri Poincare (1854–1912). The specific title of his book in French is "Mecanismes communs aux phénomènes disparates", Paris 1921. Also, titles of two his first books in Serbian are very important (see Refs [11, 12]).

It is hoped that this paper will contribute to a broader understanding of mathematical phenomenology and phenomenological mappings and will further promote the application of nonlinear functional spaces and transformations in diverse scientific disciplines.

Linear and nonlinear transformations are important basic elements in tensor calculus and applications of tensor calculus (see Ref. [13]).

Starting from the notion of linear and nonlinear transformations, affine and functional-nonlinear mappings of coordinates and coordinate systems, geometrical and kinematical invariants along linear or nonlinear transformations their coordinates from one to other coordinate system are pointed out in Ref. [13].

In a curvilinear coordinate system, coordinates of a geometrical or kinematical point are not equal as coordinates of its corresponding position vector. Expressions of basic vectors of tangent space of kinematical point vector positions in generalized curvilinear coordinate systems for the cases of orthogonal and nonorthogonal curvilinear coordinate systems are derived and presented in Ref [13]. Examples of expressions of basic vectors of tangent space of kinematical point vector position in polar-cylindrical, spherical, parabolic-cylindrical and three-dimensional-thee-parabolic system of curvilinear orthogonal coordinates are, also, presented. Next, expressions of change of basic vectors of tangent space of kinematical point vector position with time, also, are done. Geometrical (physical), covariant and contra-variant coordinates of position vector of a kinetic mass point, in a coordinate system determined by basic vectors of tangent space of this kinetic point vector position, in generalized curvilinear coordinate systems, are pointed out and determined. Original expressions of angular velocity and velocity of dilatations of basic vectors of tangent space of kinetic point vector position, in generalized curvilinear coordinate systems as well as in series of special orthogonal curvilinear coordinate system are derived by author of this paper and presented.

2. Applications of Tensor Calculus in the Mechanics of a Material Point

We consider a material point N, a mass m, whose position in space at a given moment in time t is determined by a position vector $\vec{r}(t)$.

In order to define the essential concepts of the mechanics of a material point, we introduce the following basic definitions: the definition of velocity $\vec{v}(t)$ and the definition of momentum, as well as the definition of acceleration and the definition of inertial force (see References [3, 6, 14].

Definition 1. The velocity $\vec{v}(t)$ of a material point is the limit value of the ratio of the distance between two of its close positions N_t and $N_{t+\Delta t}$ the specified position vectors $\vec{r}(t)$ and $\vec{r}(t+\Delta t)$ and the time interval Δt, for which that material point moves from one to another, when that time interval tends to zero:

$$\vec{v} \stackrel{\text{def}}{=} \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t)-\vec{r}(t)}{\Delta t} = \frac{d\vec{r}}{dt} \tag{1}$$

This means that the velocity $\vec{v}$ of a material point at an instant in time t is by definition equal to the natural derivative of the position vector $\vec{r}(t)$ with respect to time t.

Velocity is a vector and as such is invariant.

The definition of velocity $\vec{v}$ establishes a relationship between distance and time. The dimension of the intensity of velocity is: $dim\ \ v = LT^{-1}$, where L we denote the dimension of length, whose unit is the centimeter cm of length or meter, and the dimension of time T, whose unit is second. The unit of velocity is ms^{-1}.

Definition 2. The momentum $\vec{p}(t)$ of a material point at some instant of time t is the product of its mass m and its velocity vector $\vec{v}(t)$ at that instant of time t:

$$\vec{p} \stackrel{\text{def}}{=} m\ \vec{v} \tag{2}$$

This means that the linear momentum $\vec{p}(t)$ of a material point is a vector and as such is *invariant.*

By defining the momentum of a material point, we have established the relationship between mass, distance, and time. The dimension of the intensity of the momentum of a material point is: $dim\, p = MLT^{-1}$, where M we denoted the dimension of mass, whose unit is gram of mass g or kg kilogram of mass, with we denoted the dimension of length, whose unit is centimeter of length cm or meter, and with T the dimension of time, whose unit is second. The unit of momentum is (Ns).

Definition 3. The acceleration $\vec{a}(t)$ of a material point is the limiting value of the ratio of the difference of its velocities $\vec{v}(t)$ and $\vec{v}(t+\Delta t)$ in two close positions N_t and $N_{t+\Delta t}$,, determined by the position vectors $\vec{r}(t)$ and $\vec{r}(t+\Delta t)$, and the time interval Δt, for which the material point moves from one to the other, when this time interval tends to zero:

$$\vec{a} \stackrel{\text{def}}{=} \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t} = \frac{d\vec{v}}{dt} \tag{3}$$

This means, that the acceleration $\vec{a}(t)$ of a material point at an instant of time t is, by definition, equal to the natural derivative of its velocity vector $\vec{v}(t)$ with respect to time t. Acceleration $\vec{a}(t)$ is a vector and as such is *invariant.*

The definition of acceleration $\vec{a}(t)$ establishes a relationship between distance and time. The dimension of the intensity of acceleration is: $dim\ \ a = LT^{-2}$, where L we denote the dimension of length, whose unit is a centimeter cm of length or a meter m, and the dimension of time T, whose unit is a second. The unit of acceleration $\vec{a}(t)$ is ms^{-2}.

Definition 4. The force of inertia $\vec{\boldsymbol{I}}_{\boldsymbol{F}}(t)$ of a material point at some instant of time t is the product of its mass m and the oppositely directed vector $\vec{a}(t)$ of its acceleration at that instant of time t:

$$\vec{\boldsymbol{I}}_{\boldsymbol{F}}(t) \stackrel{\text{def}}{=} -m\ \vec{a}(t) \tag{4}$$

This means that the force $\vec{\boldsymbol{I}}_{\boldsymbol{F}}(t)$ of inertia of a material point, at some instant of time t, is a vector and as such *is invariant.*

The dimension of the force $\vec{\boldsymbol{I}}_{\boldsymbol{F}}(t)$ of inertia of a material point is: $dim\ \ \boldsymbol{I}_{\boldsymbol{F}} = MLT^{-2}$, where M we denoted the dimension of mass, whose unit is gr gram of mass or kg kilogram of mass, by L denoted the dimension of length, whose unit is centimeter of length or meter, and by the dimension of time, whose unit is second. The unit of inertial force $\vec{\boldsymbol{I}}_{\boldsymbol{F}}(t)$ of a material point is $kgms^{-2}$ or N (Newton).

If we consider a material point in three-dimensional E^3 Euclidean space, the position of that point is determined by a system of three mutually ordered independent quantities $q^i,\ i = 1,2,3$, denoted as generalized coordinates, and in the general case curvilinear, coordinates of free points. Let the Descartes coordinates of the material point be $x^j,\ j = 1,2,3$. We can also determine the position of that same point using curvilinear coordinates $y^k,\ k = 1,2,3$.

Let the relationship between some generalized coordinates $q^i,\ i = 1,2,3$ and Descartes coordinates $x^j,\ j = 1,2,3$ be given by the expression:

$$x^j = x^j(q^1, q^2, q^3); \qquad Det\left(\frac{\partial x^j}{\partial q^i}\right) \neq 0; \quad i,j = 1,2,3 \tag{5}$$

then for the position vector of the material point we can write:

$$d\vec{r} = \vec{e}_j dx^j = \vec{g}_i dq^i = \vec{t}_k dy^k \qquad i,j,k = 1,2,3 \tag{6}$$

where $\vec{e}_j$ are orthonormal vectors, of unit intensity, of the orientation of the coordinate axes of the Descartes coordinate system $x^j,\ j = 1,2,3$, and they do not change orientation; $\vec{g}_i = \frac{\partial \vec{r}}{\partial q^i}$ are the basis vectors of the curvilinear generalized coordinate system $q^i,\ i = 1,2,3$; $\vec{t}_k = \frac{\partial \vec{r}}{\partial y^k}$ are the basis vectors of the curvilinear coordinate system $y^k,\ k = 1,2,3$.

According to the principle of causal determinism (see Refs [4, 5,14]), the motion of a material point is uniquely determined by its governing parameters. The corresponding physical quantities remain invariant regardless of the coordinate system used for their mathematical representation. Based on the pre-principle of invariance of motion and the properties of the motion of material points, it does not depend on the form of the statement, because a truth about motion established and written down in a certain form of a language is equally contained in a record of another form or another script. This means that we can express the defined motion of a material point in different coordinate systems, and denote it by specific real numbers for mass, time, and distance, but that all three realities are invariant (see Ref. [13]).

The square of the arc element between two positions of a material point at an instant in time t and $t + dt$ can be written as:

$$ds^2 = d\vec{r} \cdot d\vec{r} = \delta_{ij} dx^i dx^j = g_{ij} dq^i dq^j = \hat{g}_{ij} dy^i dy^j \tag{7}$$

where:

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{8}$$

Kronecker's delta symbols;

$$g_{ij} = \vec{g}_i \cdot \vec{g}_j = \frac{\partial \vec{r}}{\partial q^i} \cdot \frac{\partial \vec{r}}{\partial q^j} = \delta_{kl} \frac{\partial x^k}{\partial q^i} \frac{\partial x^l}{\partial q^j} \tag{9}$$

The previous expressions (9) are the coordinates of the basic metric tensor in E^3, with respect to the generalized coordinates $q^i,\ i = 1,2,3$;

$$\hat{g}_{ij} = \vec{t}_i \cdot \vec{t}_j = \frac{\partial \vec{r}}{\partial y^i} \cdot \frac{\partial \vec{r}}{\partial y^j} = \delta_{kl} \frac{\partial x^k}{\partial y^i} \frac{\partial x^l}{\partial y^j} \tag{10}$$

The previous expressions (10) are the coordinates of the basic metric tensor in curvilinear tree dimensional coordinate system, with respect to the generalized coordinates $y^k,\ k = 1,2,3$.

2.1 The coordinates of the position vector in the generalized coordinate system.

Now the position vector of the material point can be written in the form:

$$\vec{r} = \vec{e}_j x^j = \vec{g}_i r^i \quad i,j = 1,2,3 \tag{11}$$

where $\vec{e}_j$ are orthonormal vectors, of unit intensity, of the orientations of the coordinate axes of the Descartes coordinate system of coordinates $x^j,\ j = 1,2,3$, and they do not change orientation; $\vec{g}_i = \frac{\partial \vec{r}}{\partial q^i}$ are the basis vectors of the curvilinear generalized coordinate system with coordinates $q^i,\ i = 1,2,3$; while r^i are the contravariant coordinates of the position vector. Let:

$$\vec{r}(q^1, q^2, q^3) = \vec{e}_j x^j(q^1, q^2, q^3) = \vec{g}_i r^i(q^1, q^2, q^3) \quad i,j = 1,2,3 \tag{12}$$

By differentiating (12) with respect to the corresponding curvilinear coordinate $q^i,\ i = 1,2,3$, we obtain that the vectors $\vec{g}_i,\ i = 1,2,3$ are functions of the position of the material

point and in general these vectors at each point N in space form a triad of generalized coordinates - a vector basis for generalized coordinates $q^i,\ i = 1,2,3$:

$$\vec{g}_i(q^1,q^2,q^3) = \frac{\partial \vec{r}(q^1,q^2,q^3)}{\partial q^i} = \vec{e}_j \frac{\partial x^j(q^1,q^2,q^3)}{\partial q^i},\ i,j = 1,2,3 \tag{13}$$

and these vectors are neither of constant intensity, nor do they always form the same angles at every point in space, but they continuously change in intensity and direction from point to point, by moving the material point through a series of successive positions in space.
By differentiating (12) in terms of the Descartes coordinate coordinates $x^j,\ j = 1,2,3$, respectively, we obtain the relation of the basis vectors $\vec{e}_j,\ j = 1,2,3$ to the basis vectors $\vec{g}_i,\ i = 1,2,3$ of the generalized coordinate triad $q^i,\ i = 1,2,3$, such that:

$$\vec{e}_j(q^1,q^2,q^3) = \frac{\partial \vec{r}(q^1,q^2,q^3)}{\partial q^i}\frac{\partial q^i(x^1,x^2,x^3)}{\partial x^j} = \vec{g}_i \frac{\partial q^i(x^1,x^2,x^3)}{\partial x^j};i,j = 1,2,3 \tag{14}$$

Based on expressions (13) and (14), we can write relation (12) in the following form:

$$\vec{r}(x^1,x^2,x^3) = \vec{g}_i x^j \frac{\partial q^i(x^1,x^2,x^3)}{\partial x^j} = \vec{g}_i r^i(x^1,x^2,x^3),i,j = 1,2,3 \tag{15}$$

respectively

$$\vec{r}(q^1,q^2,q^3) = \vec{e}_j x^j(q^1,q^2,q^3) = \vec{e}_j r^i(q^1,q^2,q^3)\frac{\partial x^j(q^1,q^2,q^3)}{\partial q^i} \tag{16}$$

From the last relations we get:

$$r^i(x^1,x^2,x^3) = x^j \frac{\partial q^i(x^1,x^2,x^3)}{\partial x^j},\ i,j = 1,2,3 \tag{17}$$

$$x^j(q^1,q^2,q^3) = r^i(q^1,q^2,q^3)\frac{\partial x^j(q^1,q^2,q^3)}{\partial q^i},\ i,j = 1,2,3 \tag{18}$$

The quantities $r^i(x^1,x^2,x^3)$ determined by expression (17) represent the covariant coordinates of the position vector with respect to the system of curvilinear generalized coordinates $q^i,\ i = 1,2,3$. It should be noted here that the generalized coordinates $q^i,\ i = 1,2,3$ of a point N and the coordinates $r^i(x^1,x^2,x^3)$ of the position vector of that same point are not the same, as is the case with the Descates coordinates of a point and the coordinates of its position vector.

2.2 The velocity of a material point in tensor form.

In vector form, the velocity of a material point is determined by definition in the form:

$$\vec{v} \stackrel{\text{def}}{=} \frac{d\vec{r}}{dt} = \frac{\partial \vec{r}}{\partial q^i}\frac{dq^i}{dt} = \vec{g}_i \dot{q}^i = \frac{d}{dt}(r^i \vec{g}_i) = \frac{dr^i}{dt}\vec{g}_i + r^i \frac{d\vec{g}_i}{dt} = \left(\frac{dr^i}{dt} + \omega^i_j r^j\right)\vec{g}_i = v^i \vec{g}_i \tag{19}$$

where we introduced the notation for the rotation tensor in the expression for the time derivative of the basis vectors $\vec{g}_i,\ i = 1,2,3.$:

$$\frac{d\vec{g}_i}{dt} = \omega^j_i \vec{g}_j = \frac{\partial \vec{g}_i}{\partial q^j}\dot{q}^j = \Gamma^k_{ij}\vec{g}_k \dot{q}^j \tag{20}$$

as it is:

$$\frac{\partial \vec{g}_i}{\partial q^j} = \Gamma^k_{ij}\vec{g}_k \tag{21}$$

It follows that the coordinates of the rotation tensor ω^i_j are:

$$\omega^j_i = \Gamma^j_{ik}\dot{q}^k \tag{22}$$

The rotation tensor coordinates have the dimensions $dim\,\omega^j_i = T^{-1}$, and their unit is sec^{-1}, which is the unit of angular velocity, circular frequency, or incidence.
The covariant coordinates $v^i,\ i = 1,2,3$ of the velocity vector $\vec{v}$ are:

$$v^i = \frac{dr^i}{dt} + \omega^i_j r^j, \quad i,j = 1,2,3. \tag{23}$$

The velocity of a material point may also be expressed as:

$$\vec{v} \stackrel{\text{def}}{=} \frac{d\vec{r}}{dt} = \frac{dr^i}{dt}\vec{g}_i + r^i \frac{d\vec{g}_i}{dt} = \left(\frac{\partial r^i}{\partial q^j} + r^k \Gamma^i_{kj}\right)\dot{q}^j \vec{g}_i = v^i \vec{g}_i \tag{24}$$

from which it follows that the coordinates of the velocity vector in any system of curvilinear coordinates can be written in the form:

$$v^i = \frac{Dr^i}{dt} = \frac{dr^i}{dt} + r^k \Gamma^i_{kj}\dot{q}^j = \left(\frac{\partial r^i}{\partial q^j} + r^k \Gamma^i_{kj}\right)\dot{q}^j \tag{25}$$

where $\frac{Dr^i}{dt}$ the natural derivatives of the coordinates of the position vector with respect to time. Now we denote by

$$\nabla_j r^i = \frac{\partial r^i}{\partial q^j} + r^k \Gamma^i_{jk} \tag{26}$$

That is the covariant derivative ∇_j of the contravariant coordinates r^i of the position vector $\vec{r}$ of a material point.

Intensity and the square of the velocity of a material point. The square of the velocity of a material point is a scalar invariant and can be written in the form

$$2(v)^2 = \left(\frac{\partial \vec{r}}{\partial q^i} \cdot \frac{\partial \vec{r}}{\partial q^j}\right)\dot{q}^i \dot{q}^j = g_{ij}\dot{q}^i \dot{q}^j = g_{ij}\frac{Dr^i}{dt}\frac{Dr^j}{dt} \tag{27}$$

The covariant coordinate $r_j,\ j = 1,2,3$ of the position vector $\vec{r}$ of a material point is obtained as the scalar product of the position vector $\vec{r}$ and the basis vector $\vec{g}_j$, as:

$$r_j = \left(\vec{r} \cdot \vec{g}_j\right) = \left(\vec{r} \cdot \frac{\partial \vec{r}}{\partial q^j}\right) \tag{28}$$

Let us now determine the partial derivative of the covariant coordinate $r_j,\ j = 1,2,3$ of the position vector $\vec{r}$ of the material point in generalized coordinates $q^i,\ i = 1,2,3$, based on which we obtain:

$$\frac{\partial r_j}{\partial q^i} = \left(\frac{\partial \vec{r}}{\partial q^i} \cdot \vec{g}_j\right) + \left(\vec{r} \cdot \frac{\partial \vec{g}_j}{\partial q^i}\right) = \left(\vec{g}_i \cdot \vec{g}_j\right) + \left(\vec{r} \cdot \Gamma^k_{ji}\vec{g}_k\right) = g_{ij} + \Gamma^k_{ji} r_k \tag{29}$$

because $\frac{\partial \vec{g}_i}{\partial q^j} = \Gamma^k_{ij}\vec{g}_k$. From the previous relation (28) we can show that:

$$\nabla_i r_j = \frac{\partial r_j}{\partial q^i} - r_k \Gamma^k_{ji} = g_{ij} \tag{30}$$

and this means that: the covariant derivative $\nabla_i r_j$ of the covariant coordinate r_j of the position vector $\vec{r}$ of the material point in terms of the generalized coordinate q^i is equal, by the corresponding index, to the coordinate of the metric tensor.

Basis vectors $\vec{g}_i,\ i = 1,2,3$ have the property that they are covariantly constant. We prove this in the following way: according to (20) it follows that:

$$\frac{d\vec{g}_i}{dt} - \frac{\partial \vec{g}_i}{\partial q^j}\dot{q}^j = \frac{\partial \vec{g}_i}{\partial q^j}\dot{q}^j - \Gamma^k_{ij}\vec{g}_k \dot{q}^j = \nabla_j \vec{g}_i \dot{q}^j = 0 \tag{31}$$

It also follows that:

$$\nabla_j \vec{g}_i = \frac{\partial \vec{g}_i}{\partial q^j} - \Gamma^k_{ij}\vec{g}_k = 0 \tag{32}$$

This means that the covariant derivatives of the basis vectors $\vec{g}_j,\ j = 1,2,3$in the generalized curvilinear coordinates $q^i,\ i = 1,2,3$ of the curvilinear coordinate system are equal to zero.

2.3 The linear momentum of a material point in tensor form.

Since, by definition, the linear momentum $\vec{p}$ of a material point is defined by expression (2), and the velocity by expressions (23), (24) or (25), we can write the following:

$$\vec{p} \stackrel{\text{def}}{=} m\vec{v} = m\frac{d\vec{r}}{dt} = m\left(\frac{dr^i}{dt}\vec{g}_i + r^i\frac{d\vec{g}_i}{dt}\right) = m\left(\frac{\partial r^i}{\partial q^j} + r^k\Gamma^i_{kj}\right)\dot{q}^j\vec{g}_i = mv^i\vec{g}_i \quad (33)$$

We can also write:

$$\vec{p} \stackrel{\text{def}}{=} m\vec{v} = m\vec{g}_i\dot{q}^i \quad (34)$$

The covariant coordinates $p_i,\ i = 1,2,3$ of the linear momentum of the material point in the generalized curvilinear coordinate system $q^i, i = 1,2,3$, based on relation (34), can be written in the form:

$$p_i = (\vec{p}\cdot\vec{g}_i) = m(\vec{g}_j\cdot\vec{g}_i)\dot{q}^j = mg_{ij}\dot{q}^j = a_{ij}\dot{q}^j \quad (35)$$

In the last expression (35) we introduced the notation

$$a_{ij} = m(\vec{g}_j\cdot\vec{g}_i) = mg_{ij} \quad (36)$$

which represents the covariant coordinates of the inertia tensor of a material point in a generalized coordinate system. These coordinates are moments of mass and have the dimensions of mass if the generalized coordinates have the dimension of length, but if the generalized coordinates are angles, then the coordinates of the inertia tensor can have the dimensions of mass times the length squared.

2.4 Acceleration $\vec{a}$ of a material point in tensor form.

In vector form, the acceleration $\vec{a}$ of a material point is determined by definition (3) in the form:

$$\vec{a} \stackrel{\text{def}}{=} \frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{g}_i\dot{q}^i) = \vec{g}_i\ddot{q}^i + \dot{q}^i\frac{d\vec{g}_i}{dt} = \ddot{q}^i\vec{g}_i + \frac{\partial\vec{g}_i}{\partial q^j}\dot{q}^i\dot{q}^j \quad (37)$$

and taking into account relations (20) and (21) for the acceleration of a material point we can write:

$$\vec{a} \stackrel{\text{def}}{=} \frac{d\vec{v}}{dt} = \ddot{q}^i\vec{g}_i + \frac{\partial\vec{g}_i}{\partial q^j}\dot{q}^i\dot{q}^j = \left(\ddot{q}^i + \Gamma^i_{jk}\dot{q}^j\dot{q}^k\right)\vec{g}_i \quad (38)$$

The contravariant coordinates $a^i,\ i = 1,2,3$ of the acceleration vector $\vec{a}$ in the generalized curvilinear coordinate system $q^i,\ i = 1,2,3$are:

$$a^i = (\vec{a}\cdot\vec{g}^i) = \left(\ddot{q}^l + \Gamma^l_{jk}\dot{q}^j\dot{q}^k\right)(\vec{g}^i\cdot\vec{g}_l) = \left(\ddot{q}^l + \Gamma^l_{jk}\dot{q}^j\dot{q}^k\right)\delta^i_l = \ddot{q}^i + \Gamma^i_{jk}\dot{q}^j\dot{q}^k \quad (39)$$

The acceleration vector $\vec{a}$ of a material point can be written using the coordinates of the velocity vector (24) and in the following way:

$$\vec{a} \stackrel{\text{def}}{=} \frac{d\vec{v}}{dt} = \frac{d}{dt}(v^i\vec{g}_i) = \frac{dv^i}{dt}\vec{g}_i + v^i\frac{d\vec{g}_i}{dt} = \left(\frac{\partial v^i}{\partial q^j} + v^k\Gamma^i_{kj}\right)v^j\vec{g}_i \quad (40)$$

from which it follows that the contravariant coordinates $a^i,\ i = 1,2,3$ of the acceleration vector $\vec{a}$ in any system of curvilinear coordinates can be written in the form:

$$a^i = \frac{Dv^i}{dt} = \frac{dv^i}{dt} + \Gamma^i_{kj}v^jv^k = \left(\frac{\partial v^i}{\partial q^j} + v^k\Gamma^i_{kj}\right)v^j \quad (41)$$

where $\frac{Dv^i}{dt}$ are the natural derivatives of the contravariant coordinates $v^i,\ i = 1,2,3$ of the velocity vector with respect to time.

The covariant coordinates $a_i,\ i = 1,2,3$ of the acceleration vector $\vec{a}$ in any curvilinear coordinate system $q^i, i = 1,2.3$ can also be written in terms of the square of the velocity $(v)^2$ in the form:

$$a_i = \frac{d}{dt}\frac{\partial\left(\frac{(v)^2}{2}\right)}{\partial \dot{q}^i} - \frac{\partial\left(\frac{(v)^2}{2}\right)}{\partial q^i} \tag{42}$$

We can arrive at the previous expression in the following way: We differentiate the square of the velocity $(v)^2 = (\vec{v}\cdot\vec{v})$ by the derivative of the generalized coordinate $\dot{q}^i$ with respect to time, and we obtain:

$$\frac{\partial\left(\frac{(v)^2}{2}\right)}{\partial \dot{q}^i} = \left(\vec{v}\cdot\frac{\partial\vec{v}}{\partial \dot{q}^i}\right) = \left(\vec{v}\cdot\frac{\partial\vec{r}}{\partial q^i}\right) = (\vec{v}\cdot\vec{g}_i) \tag{43}$$

as it is:

$$\frac{\partial\vec{v}}{\partial \dot{q}^i} = \frac{\partial\vec{r}}{\partial q^i} = \vec{g}_i \tag{44}$$

Now, we differentiate the relation (43) with respect to time and we get:

$$\frac{d}{dt}\frac{\partial\left(\frac{(v)^2}{2}\right)}{\partial \dot{q}^i} = \frac{d}{dt}\left(\vec{v}\cdot\frac{\partial\vec{r}}{\partial q^i}\right) = (\vec{a}\cdot\vec{g}_i) + \left(\vec{v}\cdot\frac{\partial\vec{v}}{\partial q^i}\right) \tag{45}$$

as it is:

$$\frac{d}{dt}\frac{\partial\vec{r}}{\partial q^i} = \frac{\partial\vec{v}}{\partial q^i} \tag{46}$$

From the last term of relation (45) we now obtain:

$$a_i = (\vec{a}\cdot\vec{g}_i) = \frac{d}{dt}\frac{\partial\left(\frac{(v)^2}{2}\right)}{\partial \dot{q}^i} - \left(\vec{v}\cdot\frac{\partial\vec{v}}{\partial q^i}\right) = \frac{d}{dt}\frac{\partial\left(\frac{(v)^2}{2}\right)}{\partial \dot{q}^i} - \frac{\partial\left(\frac{(v)^2}{2}\right)}{\partial q^i} \tag{47}$$

which represents proof of relation (42) by which the covariant coordinates $a_i,\ i = 1,2,3$ of the acceleration vector $\vec{a}$ are expressed in terms of the square of the velocity $(v)^2$ of a moving, in time, material point.

2.5 The inertial force $\vec{\mathbf{I}}_{\mathbf{F}}(t)$ of a material point at some instant of time t.

Since the inertial force is defined by $\vec{\boldsymbol{I}}_{\boldsymbol{F}}(t) \stackrel{\text{def}}{=} -m\ \vec{a}(t)$ relation (4) as $\vec{\boldsymbol{I}}_{\boldsymbol{F}}(t) \stackrel{\text{def}}{=} -m\ \vec{a}(t)$, then using the previously derived expressions for acceleration and its covariant and contravariant coordinates, (38), (39), (41) and (42), we can write for the inertial force $\vec{\boldsymbol{I}}_{\boldsymbol{F}}(t)$ and its coordinates: The inertial force $\vec{\boldsymbol{I}}_{\boldsymbol{F}}(t)$ in vector form:

$$\vec{\boldsymbol{I}}_{\boldsymbol{F}} \stackrel{\text{def}}{=} -m\vec{a} = -m\frac{d\vec{v}}{dt} = -m\left(\ddot{q}^i + \Gamma^i_{jk}\dot{q}^j\dot{q}^k\right)\vec{g}_i \tag{48}$$

and its contravariant coordinates in the generalized curvilinear coordinate system $q^i,\ i = 1,2.3$ are:

$$\boldsymbol{I}^i_{\boldsymbol{F}} = \left(\vec{\boldsymbol{I}}_{\boldsymbol{F}}\cdot\vec{g}^i\right) = -m(\vec{a}\cdot\vec{g}^i) = -m\left(\ddot{q}^i + \Gamma^i_{jk}\dot{q}^j\dot{q}^k\right) \tag{49}$$

or inertial force in vector form:

$$\vec{\boldsymbol{I}}_{\boldsymbol{F}} \stackrel{\text{def}}{=} -m\vec{a} = -m\frac{d}{dt}(v^i\vec{g}_i) = -m\left(\frac{\partial v^i}{\partial q^j} + v^k\Gamma^i_{kj}\right)\overrightarrow{v^j g_i} \tag{50}$$

and its contravariant coordinates in the generalized curvilinear coordinate system:

$$\mathbf{I}^i_{\mathbf{F}} = -ma^i = -m\frac{Dv^i}{dt} = -m\left(\frac{dv^i}{dt} + \Gamma^i_{kj}v^jv^k\right) = -m\left(\frac{\partial v^i}{\partial q^j} + v^k\Gamma^i_{kj}\right)v^j \tag{51}$$

and finally, the covariant coordinates of the inertia force in the generalized curvilinear coordinate system are:

$$\boldsymbol{I}_{\boldsymbol{F}i} = -ma_i = -m\left[\frac{d}{dt}\frac{\partial\left(\frac{(v)^2}{2}\right)}{\partial\dot{q}^i} - \frac{\partial\left(\frac{(v)^2}{2}\right)}{\partial q^i}\right] = -\frac{d}{dt}\frac{\partial\left(m\frac{(v)^2}{2}\right)}{\partial\dot{q}^i} - \frac{\partial\left(m\frac{(v)^2}{2}\right)}{\partial q^i} \tag{52}$$

or through the kinetic energy $2\boldsymbol{E}_K = m(v)^2$ of the material point, we can write these covariant coordinates in the form:

$$\boldsymbol{I}_{\boldsymbol{F}i} = -\frac{d}{dt}\frac{\partial(\boldsymbol{E}_K)}{\partial\dot{q}^i} - \frac{\partial(\boldsymbol{E}_K)}{\partial q^i} \tag{53}$$

3. Application of the Tensor Calculus in the Theory of Elasticity and Continuum Mechanics

3.1. Basic concepts of the model of the state of stress and the state of deformation of a deformable body

Stress tensor - is a second-order tensor whose elements are the normal stress components of tension or compression and the shear stress components at some point of a stressed body, and for three coordinate surfaces, or the plane through that point. The stress tensor can be represented using physical coordinates, or using coordinates in contravariant form σ^{nm}, covariant form σ_{nm}, or mixed form σ_n^m.

The stress tensor is associated with a reference point of a stressed body and, in general, varies from point to point of a stressed body. The stress tensor can be represented by a matrix associated with a point of a stressed body, and its columns are composed of the column matrices of total stresses $\vec{p}_n$ for the corresponding coordinate surface, or plane, through that point. In the form of a total stress vector $\vec{p}_n$, which depends on the type of external load and the geometric properties of the stressed body, as well as the points of the body and the surface (or its tangent plane), or the plane with the corresponding normal oriented unit vector $\vec{n} = n_k\vec{g}^k$. The stress tensor can be represented as a vector associated with a point on a stressed body and the surface through it, using the corresponding vectors of total stresses $\vec{p}_k$ at that point for the coordinates of the surface with normal $\vec{g}^k$ through that point in the form (see References [4, 5, 7]):

$$\vec{p}_{(n)} = \sqrt{g^{kk}}\vec{p}_{\vec{g}_k}n_k \tag{54}$$

which represents the generalized Chauchy equations. On a contour surface, these equations translate into Cauchy boundary or contour conditions. As the total stress vector for a coordinate surface can be represented by three components in one of the above forms, physical coordinates, or by coordinates in contravariant form σ^{nm}, or covariant form σ_{nm}, or mixed form σ_n^m, the previous Cauchy equations can be written in the form:

$$\vec{p}_n = \sigma^{km}n_k\vec{g}_m \tag{55}$$

The relationship between the covariant σ_{nm} and contravariant σ^{nm} coordinates of the stress tensor $\vec{p}_k$ is:

$$\sigma_{km} = g_{kl}g_{mj}\sigma^{lj} \tag{56}$$

The Navier equations of dynamic equilibrium (including rest) of a stressed deformable body subjected to volumetric forces are:

$$\sigma^{km}{}_{,k}\vec{g}_m + \rho\vec{F}_V = \rho\vec{w} \;\text{ or }\; \sigma^{km}{}_{,k}\vec{g}_m + \rho F^m\vec{g}_m = \rho\frac{\partial^2\vec{s}}{\partial t^2} \tag{57}$$

Or in the form:

$$\sigma^{km}{}_{,k}\vec{g}_m + \rho F^m\vec{g}_m = \rho\frac{\partial^2}{\partial t^2}(u^m\vec{g}_m) \tag{58}$$

Or in the form:

$$\sigma^{km}{}_{,k}\vec{g}_m + \rho F^m \vec{g}_m = \rho \frac{\partial^2 u^m}{\partial t^2}\vec{g}_m \tag{59}$$

Or in the form:

$$\sigma^{km}{}_{,k} + \rho F^m = \rho \ddot{u}^m \tag{60}$$

where the notation for the covariant derivative of the contravariant stress tensor is introduced:

$$\sigma^{ij}{}_{,j} = \frac{\partial \sigma^{ij}}{\partial x^j} + \Gamma^j_{jm}\sigma^{mi} + \Gamma^i_{mj}\sigma^{ji} \tag{61}$$

where Γ^m_{ij} Christoffel symbols are of the second type.

The relationship between the physical $\sigma_{(ij)}$ and contravariant σ^{ij} coordinates of the stress tensor, which is of second order in an orthogonal curvilinear coordinate system, is given by:

$$\sigma^{ij} = \frac{1}{A_i A_j}\sigma_{(ij)} \tag{62}$$

using Lamé coefficients A_i:

$$A_i = |\vec{g}_i| = \sqrt{(\vec{g}_i, \vec{g}_i)} = \sqrt{\left(\frac{\partial \vec{r}}{\partial x^i}, \frac{\partial \vec{r}}{\partial x^i}\right)} = \sqrt{g_{ii}} = \sqrt{\left(\frac{\partial x}{\partial x^1}\right)^2 + \left(\frac{\partial y}{\partial x^2}\right)^2 + \left(\frac{\partial z}{\partial x^3}\right)^2} \tag{63}$$

The strain tensor is a second-order tensor whose elements represent a measure of the deformation of a stressed deformable body based on a comparison of line elements from the configuration of the stressed and deformed body and the reference configuration of the natural state, and by determining the difference in the square of the lengths of their line elements. Depending on whether we express this tensor in the system of curvilinear coordinates of the reference configuration of the natural state or the reference configuration of the stressed and deformed state, we obtain two relative deformation tensors, namely: Lagrange's symmetric, second-order, relative deformation tensor ε_{ij} and Euler's symmetric, second-order, covariant, relative deformation tensor $\bar{\varepsilon}_{ij}$, respectively. The relationship between these relative deformation tensors can be written in the form (see References [4, 5, 7]):

$$(d\bar{s})^2 - (ds)^2 = 2\varepsilon_{nk}dx^n dx^k = 2\bar{\varepsilon}_{nk}d\bar{x}^n d\bar{x}^k \tag{64}$$

In the future, we will use Lagrange's symmetric, second-order, relative deformation tensor ε_{ij}, which can be expressed using the derivative of the coordinate displacement vector $\vec{s} = u^i\vec{g}_i = u_j\vec{g}^j = u_{(i)}\vec{t}_{(i)}$ in the following form:

$$\varepsilon_{nk} = \frac{1}{2}\left[\left(\vec{g}_n, \frac{\partial \vec{s}}{\partial x^k}\right) + \left(\vec{g}_k, \frac{\partial \vec{s}}{\partial x^n}\right) + \left(\frac{\partial \vec{s}}{\partial x^k}, \frac{\partial \vec{s}}{\partial x^n}\right)\right] \tag{65}$$

Considering that the covariant derivatives of the contravariant and covariant coordinates of the displacement vector are of the form:

$$u^i_{,j} = \frac{\partial u^i}{\partial x^j} + \begin{Bmatrix} i \\ k\ j \end{Bmatrix} u^k \tag{66}$$

$$u_{i,j} = \frac{\partial u_i}{\partial x^j} - \begin{Bmatrix} k \\ i\ j \end{Bmatrix} u_k$$

(67)

where $\Gamma^k_{ij} = \begin{Bmatrix} k \\ i\ j \end{Bmatrix}$ is the Christoffel symbol of the second kind in the curvilinear coordinate system x^i. For the case of small deformations, this relative deformation tensor has the form:

$$\varepsilon_{ij} = \frac{1}{2}\left[u_{i,j} + u_{j,i}\right] = \frac{1}{2}\left[\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i} - 2\begin{Bmatrix} k \\ i\ j \end{Bmatrix} u_k\right] \tag{68}$$

The volumetric deformation at some point of a rectangular deformed body is:

$$\varepsilon_V = g^{ij}\varepsilon_{i,j} = g^{ij}\varepsilon_{ij} = \varepsilon_i^i = \frac{\partial u^i}{\partial x^i} + \begin{Bmatrix} i \\ k\ j \end{Bmatrix} u^k \tag{69}$$

The physical coordinates ε_{IJ} of the Lagrange relative deformation tensor in the curvilinear orthogonal coordinate system q^I are:

$$\varepsilon_{(ij)} = \varepsilon_{IJ} = \frac{1}{2A_I A_J}\left[\frac{\partial}{\partial q^J}(A_I u_I) + \frac{\partial}{\partial q^I}(A_J u_J) - 2A_K u_K \begin{Bmatrix} K \\ J\ I \end{Bmatrix}\right] \tag{70}$$

where the Christoffel symbols of the second type are in the form:

$$\begin{Bmatrix} i \\ i\ k \end{Bmatrix} = \Gamma_{i\ k}^{i} = g^{il}\Gamma_{ik,l} = \frac{1}{2(A_i)^2}\frac{\partial}{\partial q^k}(A_i)^2, \tag{71}$$

while $\begin{Bmatrix} i \\ j\ k \end{Bmatrix} = \Gamma_{j\ k}^{i} = 0$ they are for the orthogonal coordinate system

3.2. Extension of the Rotation Tensor of the Deformable body and vector angle of rotation of line element

In Refs [15-17] the *fundamental theorem* defined in the Truesdell-Toupin's mathematical formalisms is considered as well as analyzed in the light of Helmholtz's classical theorem of the deformable body kinematics. In this sense by using Rašković's considerations of the deformable body kinematics (see Refs. [5, 7, 8]), the vector of the pure rigid rotation of the oriented line element, with infinitesimal length, for the small deformations is defined, as well as the *vector of the total relative deformation* of the oriented line element from the referent arbitrary point. For these defined vectors connected to the deformable body point and oriented line element, the corresponding expressions expressed in the vector form as well as by using covariant, contravariant and physical coordinates are derived.

The *Helmholtz's classical theorem* of the deformable body kinematics is used by Rašković (see Refs. [5, 7, 8]). In this sense by using Rašković's consideration of the deformable body kinematics, in the Refs. [7 and 8], the vector of pure rigid rotation of the oriented line element for the small deformation, as well as the vector of the pure total relative deformation of the oriented line element from the referent arbitrary point of deformable body, are defined.

In the arbitrary point $N$of the deformable body, the line element $d\vec{r}$ is oriented by unit vector $\vec{n} = \lambda^k \vec{g}_k$ where λ^k $(k = 1,2,3)$ are contravariant coordinates or direction cosine in the generalized curvilinear coordinate system, and $\vec{g}^{\,k}$ contravariant basic coordinate vectors. For oriented line element $d\vec{r}$ we can write:

$$d\vec{r} = \vec{n}dr = \vec{g}_k dx^k = \vec{g}_k \frac{dx^k}{dr} dr = \lambda^k \vec{g}_k dr \quad \text{and} \quad \frac{d\vec{r}}{dr} = \vec{n} = \lambda^k \vec{g}_k \ ,$$

where x^k are generalized curvilinear coordinates at arbitrary point N of the deformable body. Displacement vector $\vec{s}\left(x^k\right)$, with their covariant coordinates $u_i\left(x^k\right)$ is in the following form: $\vec{s} = u_i \vec{g}^{\,i}$.

Vector $\vec{\delta}_{\vec{n}}^{(N)}$ of total reduced transformation of oriented line element, with infinitesimal length, at arbitrary point N of the deformable body, from natural to deformed body state, can be defined by following formula:

$$\vec{\delta}_{\vec{n}}^{(N)} \overset{def}{=} \frac{d\vec{s}}{dr} = s^j \vec{g}_j = \left(\frac{\partial u_j}{\partial x^k} - u_i \Gamma^i_{jk}\right)\lambda^k \vec{g}^{\,j} \qquad d\vec{s} = \vec{\delta}_{\vec{n}}^{(N)} dr \tag{72}$$

This (72) introduced vector $\vec{\delta}_{\vec{n}}^{(N)}$ of a total reduced transformation of oriented line element, with infinitesimal length, at arbitrary point N of a deformable body can be decomposed on two parts in the form:

$$\vec{\delta}_{\vec{n}}^{(N)} \overset{def}{=} \frac{d\vec{s}}{dr} = \frac{d\vec{s}_{rot}}{dr} + \frac{d\vec{s}_{def}}{dr} = \vec{\delta}_{\vec{n}rot}^{(N)} + \vec{\delta}_{\vec{n}def}^{(N)}, \quad d\vec{s}_{rot} = \vec{\delta}_{\vec{n}rot}^{(N)} dr, \quad d\vec{s}_{def} = \vec{\delta}_{\vec{n}def}^{(N)} dr \tag{73}$$

The first component $\vec{\delta}_{\vec{n}rot}^{(N)}$ in previous expression (73) corresponds to pure rotation of an oriented line element at the point N around axis oriented by $\vec{\varphi}^{(N)} = \frac{1}{2} rot\vec{s}$, as if the line element was rigid. And the second component $\vec{\delta}_{\vec{n}def}^{(N)}$ in previous expression (73) corresponds to pure deformation of the oriented line element - deformation with normal strain -extension or compression and shear strain. Relative angle change in relation of rigid rotate oriented line element.

Now, it can be written:

1* for the first vector component $\vec{\delta}_{\vec{n}rot}^{(N)}$ which corresponds to pure rotation around own rotation axis, as if the line element, with infinitesimal length, was rigid (See Refs [7, 8] and Fig. 7.):

$$\vec{\delta}_{\vec{n}rot}^{(N)} = r_{jk}\lambda^k \vec{g}^{\,j} = \frac{1}{2}\left(s_{jk} - s_{kj}\right)\lambda^k \vec{g}^{\,j} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x^k} - \frac{\partial u_k}{\partial x^j}\right)\lambda^k \vec{g}^{\,j} = -r_{kj}\lambda^j \vec{g}^{\,k} \tag{74}$$

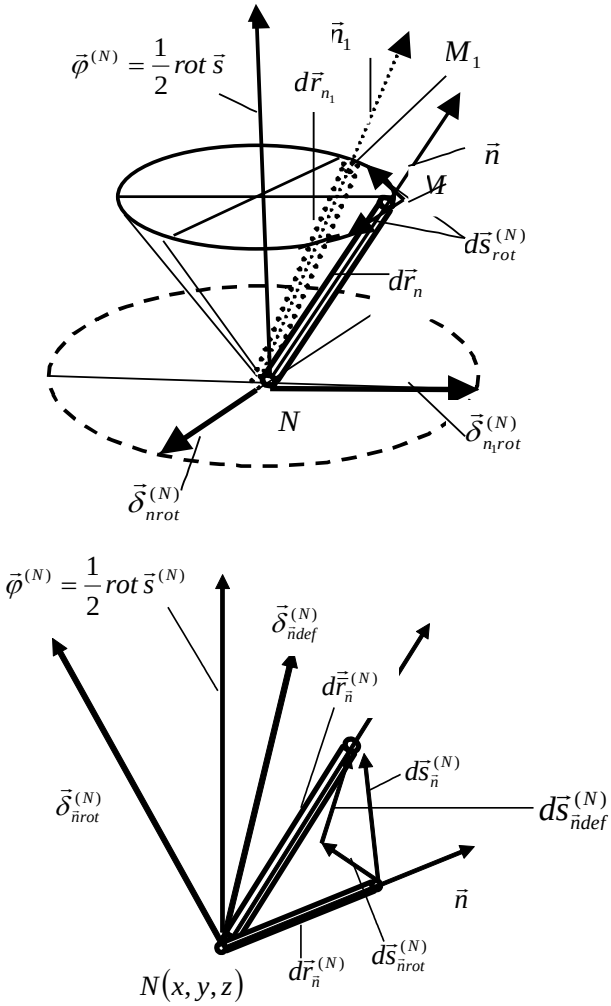


Fig. 7 Graphical representation of the vector $\vec{\delta}^{(N)}_{\vec{n}rot}$ corresponds to pure rotation of the line element and the angle $\vec{\varphi}$ of own rotation of the observed oriented line element by unit vector $\vec{n}$ at arbitrary point N of the deformable body.

2* for the second vector component $\vec{\delta}^{(N)}_{\vec{n}def}$ which corresponds to total pure relative deformation with two types of deformation: normal strain - extension or compression strain and shearing strain - change of direction angle of the oriented line element, with infinitesimal length, in relation to the other deformable body line elements or in relation to the orthogonal plane of the deviational plane of the observed oriented line element:

$$\vec{\delta}^{(N)}_{\vec{n}def} = \varepsilon_{jk}\lambda^k \vec{g}^{\,j} = \frac{1}{2}\left(s_{jk} + s_{kj}\right)\lambda^k \vec{g}^{\,j} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x^k} + \frac{\partial u_k}{\partial x^j} - 2u_i \Gamma^i_{jk}\right)\lambda^k \vec{g}^{\,j} = \varepsilon_{kj}\lambda^j \vec{g}^{\,k} \quad (75)$$

Now, let us prove that the first vector $\vec{\delta}^{(N)}_{\vec{n}rot}$ corresponds to pure rotation of the line element as if it was rigid. For that reason, it needs to be proven that it is:

$$\vec{\delta}^{(N)}_{\vec{n}rot} = \frac{1}{2}\left[rot\vec{s}, \vec{n}\right] = \left[\vec{\varphi}^{(N)}, \vec{n}\right] \quad (76)$$

Angle $\vec{\varphi}$, the angle of own rotation of the observed oriented line element, as if it was rigid, as well as of the other line elements with other orientation drown from the same observed deformable body point, is defined by the following expression:

$$\vec{\varphi}^{(N)} = \frac{1}{2} rot\vec{s} = \frac{1}{2} e^{ijk}\left(\frac{\partial u_k}{\partial x^j} - \frac{\partial u_j}{\partial x^k}\right)\vec{g}_i = \varphi^i \vec{g}_i \quad (77)$$

Axis orientation of the own rotation of the observed line element, as if it was rigid, as well as of the other line elements with other orientation drawn from the same observed deformable body point, is defined by the previous vector.

Figure 6 shows the line element $d\vec{r}_{\vec{n}}$ oriented by unit vector $\vec{n}$ and own axis of rotation oriented by vector $\vec{\varphi}^{(N)}$ connected at observed point N, as well as the corresponding vector $\vec{\delta}^{(N)}_{\vec{n}rot}$ of relative rotations as if it was rigid.

Fundamental theorem: Transformation of any oriented line element, with infinitesimal length, drown from any arbitrary deformable body point from natural undeformable state to deformable state configuration may be regarded as a result of translation of the line element, as if it was rigid, a rotation around proper own rotation axis for this point, as if the line element was rigid (rigid rotation around own rotation axis for done point), and of a pure deformation of line deformable element which is composed of two parts: the first part is normal strain as a extension-dilatation or compression-dilatation and the second is shearing strain as a change of direction angle from direction of pure rotation as if the line element was rigid (see Refs [5, 7, 8, 15-17]).

Fundamental theorem of transformation of a deformable sphere to ellipsoid: Transformation of any elementary deformable sphere, composed from line elements, *with infinitesimal length*, drown around any arbitrary deformable body point from natural undeformable state to deformable state configuration may be regarded as a result of transformation to ellipsoid in the following way: a translation of the elementary sphere as if it was rigid, a rotation of the elementary sphere around proper rotation axis for this point, as if it was a rigid elementary sphere (rigid rotation around own rotation axis for done point), and a pure deformation of elementary sphere into ellipsoid. Pure deformation of the line deformable elements which may be composed of two parts: first part is normal strain as an extension-dilatation or compression-dilatation and shearing strain as a change of direction angle from line element direction of pure rotation as if the line element was rigid.

The presented tensor-based formulations provide a unified description of several classical problems in dynamics, elasticity, and continuum mechanics. The use of vectors and tensors associated with arbitrary points and directions enables a clearer interpretation of inertia, stress, and strain states. Such formulations offer significant pedagogical and analytical advantages and reveal important analogies between different mechanical models. (see Refs [11, 12, 19-22]).

4. Basic elements of the application of tensor calculus in fracture mechanics

4.1. Energy quantity tensor

*The energy rate tens*or - (Energy rate tensor) - is defined at each point of the body using the following expression (see References [23, 24, 25]):

$$P_{jk} = W\,\delta_{jk} - \sigma_{ik} u_{i,j} \tag{78}$$

where: W is the elastic potential of the body, while are the elements of the stress tensor σ_{ik} at the observed point of the stressed body, $u_{i,j}$ are the covariant derivatives of the covariant coordinates of the displacement vectors $\vec{s}$ of the stressed and deformed body, and δ_{jk} are Kronecker's delta symbols. The energy quantity tensor is also known by the name of J. D. Eshelby (1956) as the Eshelby tensor. It can be shown how to derive an expression for the energy tensor of a deformable body.

Therefore, for example, in the coordinate system x, y, z, the elastic potential W can be determined using the expression:

$$W = \frac{1}{2}\left[\left(\vec{p}_x, \frac{\partial \vec{s}}{\partial x}\right) + \left(\vec{p}_y, \frac{\partial \vec{s}}{\partial y}\right) + \left(\vec{p}_z, \frac{\partial \vec{s}}{\partial z}\right)\right] \tag{79}$$

where: $\vec{p}_x$, $\vec{p}_y$, $\vec{p}_z$ vectors of total stress for the coordinate plane at the observed point, displacement vector $\vec{s}$; or in form:

$$W = \frac{1}{2}\left[\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz}\right] \tag{80}$$

The covariant derivatives of the covariant coordinates of the displacement vector $u_{i,j}$ of the stressed and deformed body can be written in the form:

$$u_{i,j} = \frac{\partial u_i}{\partial x^j} - \begin{Bmatrix} k \\ i\ j \end{Bmatrix} u_k \tag{81}$$

where $\Gamma_{ij}^k = \begin{Bmatrix} k \\ i\ j \end{Bmatrix}$ is the Christoffel symbol of the second kind in the curvilinear coordinate system x^i.

The energy quantity tensor, also known by the name of J. D. Eshelby (1956) as the Eshelby tensor, can be obtained starting from the Lagrangian of a particle of a deformable body in the vicinity of some reference point:

$$L = E_k - E_p \tag{82}$$

where: E_k the kinetic energy of a particle whose displacement vector is $\vec{s} = u_i \vec{g}^i$ where are the covariant coordinates u_i, $i = 1,2,3$ of the displacement vector $\vec{s} = u_i \vec{g}^i$, and $\vec{g}^i$ the basis vectors of the curvilinear coordinate system are x^i, and is equal to:

$$E_k = \frac{1}{2}\rho\left(\dot{\vec{s}}, \dot{\vec{s}}\right) = \frac{1}{2}\rho \dot{u}_i \dot{u}_j \delta_{ij} \tag{83}$$

where ρ is the specific mass - the mass density of the body; E_p the potential energy of the body particle for which we take the elastic potential of the body, which $E_p = W$ we can determine using the expression:

$$W = \frac{1}{2}\left[\left(\vec{p}_x, \frac{\partial \vec{s}}{\partial x}\right) + \left(\vec{p}_y, \frac{\partial \vec{s}}{\partial y}\right) + \left(\vec{p}_z, \frac{\partial \vec{s}}{\partial z}\right)\right]. \tag{84}$$

We can write the Lagrangian in the form:

$$L = \frac{1}{2}\rho \dot{u}_i \dot{u}_j \delta_{ij} - W\left(u_{i,j}\right) \tag{85}$$

Let us now look for the derivative of that functional in the curvilinear generalized coordinate x^i, so that we obtain:

$$\frac{dL}{dx^j} = \frac{\partial L}{\partial x^j} + \frac{\partial L}{\partial u_i} u_{i,j} + \frac{\partial L}{\partial \dot{u}_i} \dot{u}_{i,j} + \frac{\partial L}{\partial u_{i,k}} u_{i,jk} \tag{86}$$

If we introduce the assumption that these are quasi-static problems of stress in deformable bodies and taking into account that:

$$\sigma_{ij,j} - \rho \ddot{u}_i = 0 \tag{87}$$

we obtain that the derivative $\frac{\partial L}{\partial x^j}$ can be written, using expression (86) and the previous expression (87), in the form:

$$\frac{\partial L}{\partial x^j} = \frac{dL}{dx^j} - \frac{\partial L}{\partial u_{i,k}} u_{i,jk} \tag{88}$$

Now using expression (86) for the Lagrangian, the equilibrium equations (88), as well as Castigliano's formulas for the general case of the spatial state of stress of an elastic body:

$$\frac{\partial W}{\partial u_{i,j}} = \sigma_{ij} \tag{89}$$

We can write the derivation $\frac{\partial L}{\partial x^j}$ in the form:

$$\frac{\partial L}{\partial x^j} = \frac{dL}{dx^j} - \sigma_{ik} u_{i,jk} = -\frac{dW}{dx^k}\delta_{jk} + \left(\sigma_{ik} u_{i,j}\right)_{,k} \tag{90}$$

We can write the derivation in the form:

$$\frac{\partial L}{\partial x^j} = -\frac{\partial}{\partial x^k}\left(W\,\delta_{jk} - \sigma_{ik} u_{i,j}\right) \tag{91}$$

We see that the expression in parentheses of the previous partial derivative (91):

$$P_{jk} = W\,\delta_{ik} - \sigma_{ik} u_{i,j} \tag{92}$$

What is the defined *energy quantity tensor or Eshelby tensor*. Expression (92) expressed using the Eshelby tensor is simplified and reduced to the form:

$$\frac{\partial L}{\partial x^j} = -\frac{\partial P_{jk}}{\partial x^k} = -P_{jk,k} \tag{93}$$

If the energy tensor is known, the elastic potential can be calculated as:

$$W = \iint_S P_{ij} n^i x^i dS \tag{94}$$

where $\vec{n} = n^i \vec{g}_i$ is the unit vector normal to the contour surface, and $\vec{r} = x^j \vec{g}_j$ is the position vector of the reference point of the body on the contour surface S of the body.

For the case of a three-dimensional state of stress and strain, the Eshelby energy tensor is:

$$P = \begin{pmatrix} W - \sigma_{i\,1} u_{i,1} & -\sigma_{i\,2} u_{i,1} & -\sigma_{i\,3} u_{i,1} \\ -\sigma_{i\,1} u_{i,2} & W - \sigma_{i\,2} u_{i,2} & -\sigma_{i\,3} u_{i,2} \\ -\sigma_{i\,1} u_{i,3} & -\sigma_{i\,2} u_{i,3} & W - \sigma_{i\,3} u_{i,3} \end{pmatrix} \tag{95}$$

or, in the form

$$P = \begin{pmatrix} W-(\sigma_{11}u_{1,1}+\sigma_{21}u_{2,1}+\sigma_{31}u_{3,1}) & -(\sigma_{12}u_{1,1}+\sigma_{22}u_{2,1}+\sigma_{32}u_{3,1}) & -(\sigma_{13}u_{1,1}+\sigma_{23}u_{2,1}+\sigma_{33}u_{3,1}) \\ -(\sigma_{11}u_{1,2}+\sigma_{21}u_{2,2}+\sigma_{31}u_{3,2}) & W-(\sigma_{12}u_{1,2}+\sigma_{22}u_{2,2}+\sigma_{32}u_{3,2}) & -(\sigma_{13}u_{2,2}+\sigma_{23}u_{2,2}+\sigma_{33}u_{3,2}) \\ -(\sigma_{11}u_{1,3}+\sigma_{21}u_{2,3}+\sigma_{31}u_{3,3}) & -(\sigma_{12}u_{1,3}+\sigma_{22}u_{2,3}+\sigma_{32}u_{3,3}) & W-(\sigma_{13}u_{1,3}+\sigma_{23}u_{2,3}+\sigma_{33}u_{3,3}) \end{pmatrix} \tag{96}$$

If we assume that the Lagrangian of a deformable, ideally elastic body does not depend explicitly on time, we can prove that the following equation holds for the energy density $\boldsymbol{e}$:

$$\frac{\partial \boldsymbol{e}}{\partial t} = \left(\sigma_{ij}\dot{u}_i\right)_{,j} \tag{97}$$

where $\boldsymbol{e}$ is the total energy of a material particle of unit density of the material in the vicinity of the reference point; σ_{ik} are the elements of the stress tensor at the observed point of the stressed body, and $\dot{u}_i$ are the covariant coordinates of the displacement vector $\vec{s}$ of the stressed and deformed body. Also, we can show that for reversible processes, the change in the internal energy of the system is equal to the increment of the work done by the external forces $\dot{E} = \dot{\boldsymbol{A}}$.

4.2 Energy density – (energy density)

Energy density – (energy density)- is the total energy of a material particle of unit density and represents the sum of the kinetic and potential energies of a material particle of unit density of the material in the vicinity of the reference point being marked (see Refs [23, 24, 25-31]).

Let us show that for reversible processes the change in the internal energy of the system is equal to the increment of the work of the external forces $\dot{E} = \dot{\boldsymbol{A}}$ using the energy density and the Lagrangian.

Using Euler's equation for the specified potential, the Lagrangian L is obtained as follows:

$$\frac{\partial L}{\partial u_i} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{u}_i} - \frac{\partial}{\partial x^j}\frac{\partial L}{\partial u_{i,j}} = 0. \tag{98}$$

We have in mind that Castigliano's formulas (89) and taking into account *Helmholtz's theore*m on the kinematics of a deformable body, i.e. that the motion of a body as a rigid body has no effect on the Lagrangian of a deformable body, we can write the Navier equations of the dynamics of a deformable body:

$$\sigma_{ij,j} - \rho\ddot{u}_i = 0 \tag{99}$$

The total derivative of the Lagrangian is:

$$\frac{dL}{dt} = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial u_i}\dot{u}_i + \frac{\partial L}{\partial \dot{u}_i}\ddot{u}_i + \frac{\partial L}{\partial u_{i,j}}\dot{u}_{i,j} \tag{100}$$

By inserting the expression for the Lagrangian of a deformable body into the previous total derivative of the Lagrangian (100), we obtain:

$$L = \frac{1}{2}\rho\dot{u}_i\dot{u}_j\delta_{ij} - W\left(u_{i,j}\right) \tag{101}$$

and taking into account (89) and (99) we obtain:

$$\frac{\partial L}{\partial t} = \frac{dL}{dt} - \rho\dot{u}_i\ddot{u}_i + \sigma_{ij}\dot{u}_{i,j} \tag{102}$$

Bearing in mind that:

$$\rho \dot{u}_i \ddot{u}_i = \frac{dE_k}{dt} \tag{103}$$

as well as

$$\sigma_{ij}\dot{u}_{i,j} = \left(\sigma_{ij}\dot{u}_i\right)_{,j} - \sigma_{ij,j}\dot{u}_i = \left(\sigma_{ij}\dot{u}_i\right)_{,j} - \rho\ddot{u}_i\dot{u}_i \tag{104}$$

from equation (102) we obtain:

$$\frac{\partial L}{\partial t} = \frac{dL}{dt} - 2\frac{dE_k}{dt} + \left(\sigma_{ij}\dot{u}_i\right)_{,j} \tag{105}$$

The total energy of a material particle is introduced, as the sum of the kinetic and potential energy of a material particle of unit density of material in the vicinity of the reference point, denoted by $\boldsymbol{e}$:

$$\boldsymbol{e} = E_k + \Pi = E_k + E_p = -(L - 2E_k) \tag{106}$$

and is called the energy density at a given point of the deformable body.
If it is assumed that the Lagrangian does not depend explicitly on time, then from (106) the following equation is obtained for the energy density $\boldsymbol{e}$:

$$\frac{\partial \boldsymbol{e}}{\partial t} = \left(\sigma_{ij}\dot{u}_i\right)_{,j} \tag{107}$$

If we multiply the energy density $\boldsymbol{e}$ by the elementary volume dV of the deformable body, and then sum all these products over the body using the resulting equation (107), i.e. by integrating it over the entire volume of the deformable body, we can obtain:

$$\dot{E} = \frac{\partial}{\partial t}\iiint_V \boldsymbol{e}\, dV = \iiint_V \left(\sigma_{ij}\dot{u}_i\right)_{,j} dV = \iint_S \sigma_{ij} n_j \dot{u}_i dS = \dot{\boldsymbol{A}} \tag{108}$$

From the last relation (108) it follows that for reversible processes the change in the internal energy of the system is equal to the increment of the work of the external forces $\dot{E} = \dot{\boldsymbol{A}}$. When there are no thermal effects (108) represent the law of conservation of energy (the first law of thermodynamics) (see Refs [32-41]):

5 INTEGRALS OF FRACTURE MECHANICS

We now turn to presenting the basic integrals of fracture mechanics, which are most commonly used in the theory of fracture mechanics. We will list all three integrals of fracture mechanics, $\boldsymbol{J_i},\ M,\ L_i,\ i = 1,2,3$, expressed using tensor calculus.

Integrals of Fracture Mechanics - are integrals defined by the conservation law and can all be derived using the Eshelby energy tensor. The energy tensor is also known by the name of J. D. Eshelby (1956) as the Eshelby tensor. We have shown in one of the previously presented problems how to derive an expression for the energy tensor of a deformable body. Here we use a ready-made expression (78), (95) or (96) for the energy rate tensor, which is defined at each point of the body (see Refs [23- 25, 32-36, 40]). Thus, using the Eshelby energy quantity tensor P_{ij}, we define three integrals of fracture mechanics: $\boldsymbol{J_i},\ M,\ \boldsymbol{L}_i,\ i = 1,2,3$ as follows:

$$\boldsymbol{J}_i = \iint_S P_{ij} n^j dS \tag{109}$$

$$M = \iint_S \left(P_{ij}x^i - \frac{1}{2}\sigma_{ij}u^i\right) n^j dS \tag{110}$$

$$\boldsymbol{L}_i = \varepsilon_{ikl}\iint_S \left(P_j^l x^k - \frac{1}{2}\sigma_j^l u^k\right) n^j dS \tag{111}$$

The first of the defined integrals (109) is defined as a stressed deformable body in space. When we define it in the plane along a closed contour, it allows us to define the J integral,

as derived by *J. C. Rice*. It should be emphasized that the ***J*** integral was used before Rice by Eshelby (1956), Sanders (1960), and Cherepanov (1968).

For a plane problem, the elastic potential or internal energy of a body can be defined in the form:

$$W = \int_0^{\varepsilon_{\alpha\beta}} \sigma_{ij}(\varepsilon) d\varepsilon^{ij} = W\left(\varepsilon_{\alpha\beta}\right) \tag{112}$$

For closed contours, we also define the integral ***L*** and this integral is:

$$I = \int_L^{\int} \left(W\,\delta_{jk} - \sigma_{ik} u_{i,j}\right) n^j dx^k = 0 \tag{113}$$

which vanishes (see Ref. [32]). The integrals and elastic potential written in this way do not contain material constants or assumptions about the relationship between the stress and deformation states of a stressed body, so as such they can be used to determine the internal energy of deformed bodies for cases when they are elastically, elastoplastically or plastically deformed. The same applies to linear and nonlinear laws of the relationship between stress and strain states.

If the contour Γ - line is closed, then the previous integral (113) is written in the following form of the ***J*** integral:

$$\boldsymbol{J} = \int_\Gamma \left(W\,\delta_{jk} - \sigma_{ik} u_{i,j}\right) n^j dx^k = \int_\Gamma \left(W\,d\,x^2 - \sigma_{ik} u_{i,j} n^j ds\right) \neq 0 \tag{114}$$

and this J integral is independent of the path (see Refs [42-44]).

It is especially important to calculate this J integral around the crack tip. And it, the ***J*** integral, for the two-dimensional stress state reduces to:

$$\boldsymbol{J} = \int_\Gamma \left(W\,\delta_{jk} - \sigma_{ik} u_{i,j}\right) n^j dx^k = \int_\Gamma \left(W\,d\,x^2 - \sigma_{ik} u_{i,j} n^j ds\right) = \int_\Gamma (W\,d\,x^2) \neq 0 \tag{115}$$

and represents the average value of the internal energy of the body.

The ***J*** integral represents the negative derivative of the potential energy with respect to the length of the crack. Since the negative derivative of the potential energy with respect to the generalized coordinate represents the generalized force, it can be concluded that the ***J*** integral represents the generalized force on the change in the length of the crack.

The **M** integral does not depend on the surface of integration in the case of a three-dimensional body stress problem and is equal to zero for any closed surface that does not contain a singularity. Using the **M** integral, it is possible, in some cases, to determine the stress intensity factor (for example, for a semi-infinite crack).

5. FINAL CONSIDERATIONS

This paper has presented selected applications of tensor calculus in three important areas of mechanics: point mechanics, elasticity theory, and fracture mechanics. The tensor formulation of fundamental mechanical quantities, including velocity, acceleration, inertial force, stress, strain, and energy-related quantities, demonstrates the advantages of coordinate-independent mathematical descriptions and provides a unified framework for the analysis of mechanical systems.

The presented formulations highlight the role of covariant and contravariant quantities in generalized curvilinear coordinate systems and illustrate their importance in both theoretical developments and engineering applications. Particular attention has been devoted to the kinematics of deformable bodies, the representation of stress and strain states, and the tensor interpretation of fundamental fracture-mechanics integrals.

Although modern computational software enables the efficient analysis of highly complex structures, a solid understanding of the tensor foundations of mechanics remains essential for the correct interpretation of numerical results and for the development of advanced theoretical models.

The author hopes that this concise overview will encourage further studies and applications of tensor calculus in mechanics, especially among younger researchers and graduate students working in mechanics, applied mathematics, and engineering sciences.

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