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SOLVING A NON-STATIONARY HEAT CONDUCTION PROBLEM IN A WALL WITH ASYMMETRIC BOUNDARY CONDITIONS USING THE LAPLACE TRANSFORM

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Abstract. *This paper presents the application of the Laplace transform to non-stationary heat conduction in a wall with asymmetric boundary conditions. The Laplace transform, a numerical method for the inverse Laplace transform, and the Mathematica software are utilized to derive a solution to the considered problem. A 3D temperature distribution over a period of 8.3 hours was examined. The temperature profile equalized gradually, and after 8.3 hours, it became nearly linear, indicating that the system reached thermal equilibrium, as anticipated during the cooling of the wall under the specified conditions. A comparison with the findings in paper [3] showed that the temperature behavior was identical.*

Key words: *Laplace transform, Non-stationary heat conduction, Asymmetric boundary conditions*

1. INTRODUCTION

The Laplace transform method is one of the methods that can be used to determine a temperature distribution. Unsteady heat conduction in walls is an important phenomenon that can often be encountered in practice. We can find a temperature distribution in the wall if we have symmetric cooling or heating. But difficulties arise in the presence of asymmetric boundary conditions. The Laplace transform method can be used in such cases.

Book [1] describes the Laplace transform method for the solution of time-dependent heat conduction problems. In [2] different methods of solving heat conduction problems are considered. One of the presented methods is the Laplace transform that is used for solving a problem of a semi-infinite medium, initially at zero, whose surface temperature at $x = 0$ is held at the value T_S from $t = 0$ onward.

In [3] Hana Charvátová and Martin Zálešák presented a mathematical model of a non-stationary heat conduction in a solid wall. The authors solved an asymmetric problem with the boundary conditions of imperfect heat transfer by using the Laplace transform and

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MAPLE, and verified their analytical solution by numerical calculation with the COMSOL Multiphysics software. The maximum difference was about 3.5 %. Paper [4] provides a comparison of five nonlinear sequence transformations applied to the Gaver functionals for the Gaver Method of Numerical Laplace Transform Inversion. The authors concluded that the Wynn rho algorithm is the most effective among the acceleration schemes considered in their work.

Natalia Yaparova [5] considered the application of different approaches based on the Laplace and Fourier transforms to solve a boundary-value inverse heat conduction problem with a steady boundary. The author analyzed these methods and performed a computational experiment, testing the performance of the algorithms and evaluating the errors of the regularized solutions provided by each approach.

In [6] the author considered certain boundary value problems for parabolic equations. The solutions were obtained with the generalized Laplace integral transform.

In this paper we will consider non-stationary heat conduction in a wall with asymmetric boundary conditions. In the experiment, we used the Laplace transform and derived an analytical solution in the s-space. By applying the numerical method for the inverse Laplace transform using the Mathematica software [7], we obtained the solution of the considered problem and calculated temperature profiles for different times. We compared our results with the solution of the same problem in paper [3].

The results presented in this paper are part of wider research into the thermal and accumulative characteristics of facade constructions for nearly zero energy buildings. Thermal mass of a building facade represents a time-dependent property of the materials used in building construction, which can accumulate and release thermal energy [8]. In this regard, buildings can be classified as high or low thermal mass buildings [9].

2. MATHEMATICAL FORMULATION

In this case, a transient heat conduction in an isotopic material solid wall is considered. The length and width of the wall are of the order of magnitude larger than its thickness (Fig. 1). The wall has thickness δ, thermal conductivity λ, density ρ and specific heat c_p. The temperatures inside the space T_{IN} and outside T_{OUT} are constantly valued. At time $t = 0$ the temperature of the whole plate is T_0.

For the heat transfer $T(x, t)$ in the wall we have the following equation:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{a}\frac{\partial T}{\partial t}, \tag{1}$$

where $a = \lambda / (\rho c_p)$ is the thermal diffusivity; $a, \lambda, \delta, \rho, c_p > 0$; $0 \le x \le \delta$; $0 \le t$.

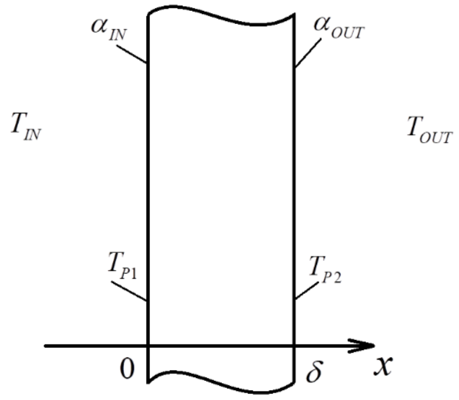

Fig. 1. Schematic of a wall

With regard to the temperature inside T_{IN} and outside T_{OUT}, the boundary conditions in our case are:

$$-\lambda \frac{\partial T(0,t)}{\partial x} = \alpha_{IN}\big(T_{IN}(t) - T(0,t)\big), \tag{2}$$

$$-\lambda \frac{\partial T(\delta,t)}{\partial x} = \alpha_{OUT}\big(T(\delta,t) - T_{OUT}(t)\big), \tag{3}$$

where are:

$T_{IN} = 15°\mathrm{C}$ - temperature in the room,

$T_{OUT} = 5°\mathrm{C}$ - temperature outside,

$T_{p_1}(0,t) = T(0,t)$ - temperature on the inside wall,

$T_{p_2}(\delta,t) = T(\delta,t)$ - temperature on the outside wall,

$\alpha_{IN}, \alpha_{OUT}$ - heat transfer coefficients on the inside and outside of the wall.

In the initial moment $t = 0$ the temperature of the whole plate is T_0, so we have the initial condition:

$$T(x,0) = T_0. \tag{4}$$

3. LAPLACE TRANSFORM

We solve equation (1) with conditions (2) – (4) by using the Laplace transform.

Let us denote the Laplace transform of the function $T(x,t)$ over the variable t by $\tilde{T}(x,s)$, i.e.

$$L_t[T(x,t)] = \int_0^\infty T(x,t)e^{-st}dt = \tilde{T}(x,s). \tag{5}$$

Then applying the Laplace transform (5) for equation (1), we get:

$$L_t\left[\frac{\partial^2 T}{\partial x^2}\right] = \frac{1}{a} L_t\left[\frac{\partial T}{\partial t}\right], \tag{6}$$

from where we derive:

$$\tilde{T}''_{x,x}(x,s) = \frac{1}{a}\Big(s\,\tilde{T}(x,s) - T(x,0)\Big), \tag{7}$$

i.e., equation (1) after using the Laplace transform (5) is:

$$\tilde{T}''_{x,x}(x,s) - \frac{1}{a}s\,\tilde{T}(x,s) = -\frac{1}{a}T_0. \tag{8}$$

Now we apply the Laplace transform to boundary conditions (2) and (3), respectively, and they become:

$$-\lambda\frac{\partial\tilde{T}(0,s)}{\partial x} = \alpha_{IN}\left(\tilde{T}_{IN}(s) - \tilde{T}(0,s)\right), \tag{9}$$

$$-\lambda\frac{\partial\tilde{T}(\delta,s)}{\partial x} = \alpha_{OUT}\left(\tilde{T}(\delta,s) - \tilde{T}_{OUT}(s)\right). \tag{10}$$

The characteristic equation for equation (8) is:

$$k^2 - \frac{1}{a}s = 0. \tag{11}$$

Its solutions are:

$$k_1 = \theta\sqrt{s}\,, \qquad k_2 = -\theta\sqrt{s}, \tag{12}$$

where:

$$\theta = \sqrt{1/a} = \sqrt{\rho\,c_p/\lambda} \tag{13}$$

The solution of the homogenous differential equation is:

$$\tilde{T}_H(x,s) = Ae^{-\theta\sqrt{s}\,x} + Be^{\theta\sqrt{s}\,x}. \tag{14}$$

Let us check if the inhomogeneous equation has a constant particular solution $\tilde{T}_P(x,s) = C$. We find that:

$$\tilde{T}_P = \frac{T_0}{s}. \tag{15}$$

The general solution is:

$$\tilde{T}(x,s) = \tilde{T}_H(x,s) + \tilde{T}_{P,} \tag{16}$$

i.e., we can get the general solution of equation (16):

$$\tilde{T}(x,s) = Ae^{-\theta\sqrt{s}\,x} + Be^{\theta\sqrt{s}\,x} + \frac{T_0}{s}. \tag{17}$$

Putting expression (17) into border conditions (9) and (10), we have:

$$-\lambda\frac{\partial\,\tilde{T}(0,s)}{\partial\,x} = \lambda\theta\sqrt{s}(A-B) = \alpha_{IN}\left(\tilde{T}_{IN}(s) - A - B - \frac{T_0}{s}\right), \tag{18}$$

$$-\lambda\frac{\partial\,\tilde{T}(\delta,s)}{\partial\,x} = \lambda\theta\sqrt{s}\big(Ae^{-\delta\,\theta\sqrt{s}} - Be^{\delta\,\theta\sqrt{s}}\big).$$

$$-\lambda\frac{\partial\,\tilde{T}(\delta,s)}{\partial\,x} = \alpha_{OUT}\left(Ae^{-\delta\,\theta\sqrt{s}} + Be^{\delta\,\theta\sqrt{s}} + \frac{T_0}{s} - \tilde{T}_{OUT}(s)\right). \tag{19}$$

from where we can get the integration constants A and B:

$$B = B(s) = \frac{\alpha_{IN} b_{11}(s) + \alpha_{OUT} b_{12}(s) e^{\delta \theta \sqrt{s}}}{s \left(c_{21}(s) e^{2\delta \theta \sqrt{s}} - c_{22}(s) \right)}, \tag{20}$$

$$A = A(s) = \frac{\alpha_{OUT} a_{11}(s) e^{\delta \theta \sqrt{s}} + \alpha_{IN} a_{12}(s) e^{2\delta \theta \sqrt{s}}}{s \left(c_{21}(s)\, e^{2\delta \theta \sqrt{s}} - c_{22}(s) \right)}, \tag{21}$$

where the constants are:

$$b_{11}(s) = \left(\lambda\theta\sqrt{s} - \alpha_{OUT}\right)\left(s\tilde{T}_{IN}(s) - T_0\right),$$

$$b_{12}(s) = \left(\lambda\theta\sqrt{s} + \alpha_{IN}\right)\left(\lambda\theta\sqrt{s} + \alpha_{OUT}\right),$$

$$c_{21}(s) = \left(\lambda\theta\sqrt{s} + \alpha_{IN}\right)\left(\lambda\theta\sqrt{s} + \alpha_{OUT}\right),$$

$$c_{22}(s) = \left(\lambda\theta\sqrt{s} - \alpha_{IN}\right)\left(\lambda\theta\sqrt{s} - \alpha_{OUT}\right),$$

$$a_{11}(s) = \left(\lambda\theta\sqrt{s} - \alpha_{IN}\right)\left(s\tilde{T}_{OUT}(s) - T_0\right),$$

$$a_{12}(s) = \left(\lambda\theta\sqrt{s} + \alpha_{OUT}\right)\left(s\tilde{T}_{IN}(s) - T_0\right).$$

Putting these constants into expression (17), we finally have:

$$\tilde{T}(x, s) = A(s) e^{-\theta\sqrt{s}\, x} + B(s) e^{\theta\sqrt{s}\, x} + \frac{T_0}{s}. \tag{22}$$

Applying the numerical method for the inverse Laplace transform given in [4], we get the solution $T(x,t)$.

4. RESULTS AND DISCUSSION

We derived the solution $T(x,t)$ of equation (1) by applying the numerical method for the inverse Laplace transform. In our case we used the following data:

$$c_p = 1200 J/(kg \cdot K), \qquad \rho = 1140 kg/m^3,$$

$$\lambda = 0.2\, W/(m \cdot K), \quad \delta = 0.1\, m,$$

$$T_{IN} = 15°\text{C}, \quad T_{OUT} = 5°\text{C}, \quad T_0 = 25°\text{C},$$

$$\alpha_{IN} = 30\, W/(m^2 \cdot K), \quad \alpha_{OUT} = 10\, W/(m^2 \cdot K).$$

We get the following graphics using the Mathematica software [7]. Fig. 2 shows a 3D temperature distribution in the wall during 30000 seconds. At the initial moment of time, temperature is 25°C in the wall. The inside temperature is 15°C and the outside temperature is 5°C according to boundary conditions (2) and (3).

Outside of the wall temperature dramatically decreases to 5°C because of the low temperature of the outside border. Inside, the temperature gradually decreases to 15°C. The temperature profile is gradually equalized, and after 8.3 hours it is a nearly straight line, which means that the system has reached thermal equilibrium.

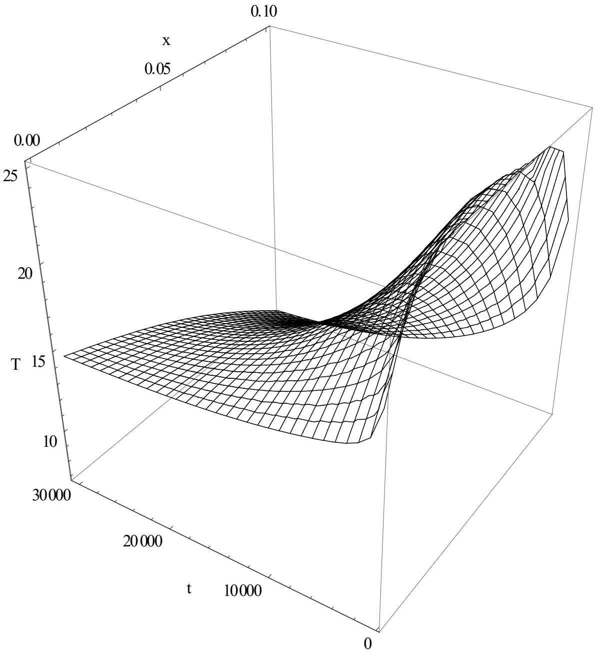

Fig. 2. 3D temperature distribution during 30000 seconds

Fig. 3 shows temperature curves for a time on the wall $t = 300 + i \cdot 1000$, $i = 0, 1, \ldots 30$. This is the same temperature behavior as in Fig. 2, but there we can clearly see how temperature depends on time. Over time, the curves become gentler, and after 6.5 hours the temperature changes very little. After 8.3 hours we can see from the graph that temperature is nearly linearly dependent on the *x* coordinate, as expected when cooling the wall under given conditions.

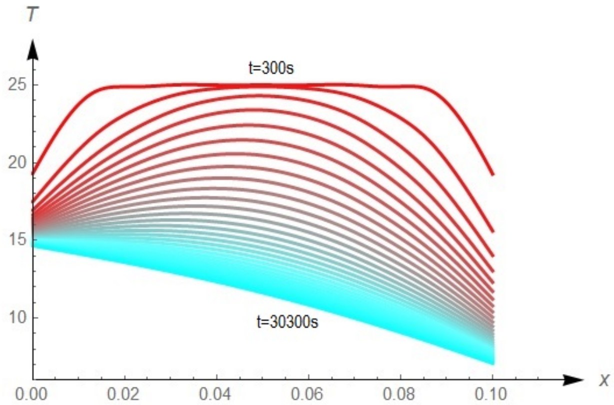

Fig. 3. Temperature curves for a time $t = 300 + i \cdot 1000$, $i = 0, 1, \ldots 30$.

To verify the Laplace transform method data we compared our results with paper [3]. The initial and boundary conditions in our paper are the same as in [3].

By comparing our results with the solution of the same problem in paper [3], we can see that temperature behaviors are almost similar.

5. CONCLUSIONS

We solved a non-stationary heat conduction problem in a wall with asymmetric boundary conditions. Using the Laplace transform method we derived an analytical solution of the transformed equation but could not get the analytical inverse Laplace transformation. So, we applied the numerical method for the inverse Laplace transform and got a numerical solution for our problem. The results are shown in Figs. 2 and 3.

By comparing our results with the solution of the same problem in paper [3], we can see that they are almost similar. This means that our solution is good enough.

In further studies, we plan to find the temperature distribution in the wall, provided that the temperature on its surfaces depends on time. We also plan to consider the effect of thermal conductivity of the wall on the temperature distribution in the wall as a function of time.

Nomenclature:

a thermal diffusivity, m^2/s
A, B constants
c_p specific heat, $J/(kg \cdot K)$
k variable
s Laplace transform variable
t time, s or h
T temperature, ℃
T_0 temperature at time $t = 0$, ℃
x coordinate distance normal to wall surface, m
$\tilde{T}(x, s)$ Laplace transform of the temperature function $T(x, t)$

Greek letters

α heat transfer coefficient, $W/(m^2 \cdot K)$
δ thickness of the wall, m
λ thermal conductivity, $W/(m \cdot K)$
ρ density, kg/m^3

Subscripts

0 initial condition
p pressure
IN inside
OUT outside
P Solution of the inhomogeneous differential equation
H Solution of the homogenous differential equation

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