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ROLLING A HEAVY BALL ON A REVOLVING SURFACE

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Abstract. *The paper defines five theorems on the properties of newly constructed orthogonal curvilinear coordinate systems on various revolving surfaces and ten theorems on the non-linear dynamics of non-slip rolling of a heavy, homogeneous and isotropic ball on revolving surfaces. Nonlinear differential equations of rolling, without sliding, a heavy, homogeneous and isotropic ball, as well as equations of phase trajectories are derived for two special cases, when the revolving surfaces are created by the rotation of a parabola, i.e., a biquadratic parabola. It is shown that for such nonlinear rolling dynamics there is a cyclic integral, as well as one cyclic coordinate in all cases of revolving surfaces. In this paper, we present only two theorems which are part of the scientific results of the research. Based on these, five theorems on curvilinear orthogonal coordinate systems constructed over surfaces of revolution and ten theorems on the properties of the dynamics of rolling a heavy ball on surfaces of revolution were defined.*

Key words: *Theorems, Revolving Surfaces, Non-linear Dynamics of Rolling of a Ball*

1. INTRODUCTION

The rolling of a ball on curvilinear paths and surfaces has attracted the attention of many researchers since ancient times, both in a scientific-theoretical approach, but also in the interest of application in engineering. There are contemporary authors who claim in their works that the rolling of a ball on a surface is a system with non-holonomic connections, and the author of this paper has shown in her works [1] that the constrains are holonomic and purely geometric, and that rolling, without slipping, a heavy homogeneous and isotropic ball on the surface is a system with two degrees of freedom of rolling and

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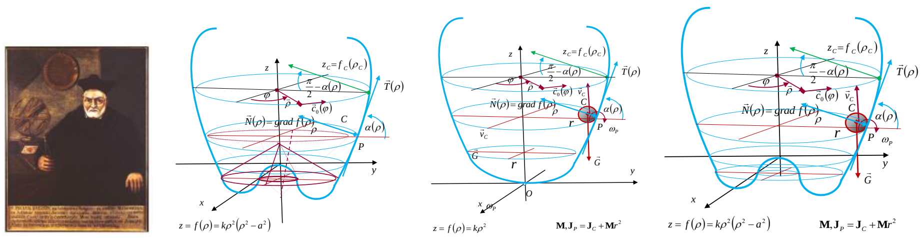
with pure geometric connections. Now let us cite one doctoral dissertation, which deals with a special and interesting complex mechanical system of rolling a ball, which contains a gyroscope [2]. The dissertation was done under the mentorship of Anton Bilimović, and defended, back in 1924, before a commission that included prominent scientists Milutin Milanković, the author of the famous work "The Canon of Sun Insolation", and Mihailo Petrović, the founder of the Serbian School of Mathematics and one of the three doctoral students of the famous scientist Julius Henri Poincaré.

The "Papus-Guldin theorems" are known, which are related to the size of the area of a revolving surface and the volume of an axisymmetric body, which is a contour surface in the form of a revolving surface. Here, we will recall the second "Papus-Guldin theorem", which defines the size of the volume of an axisymmetric body that is created by the rotation of a plane figure around an axis in its plane, which is not intersecting. It defines that the volume of a revolving body is equal to the product of the surface of the plane figure and the length of the arc of the circle described by the center of the area of the plane figure for one complete revolution around axis.

These theorems were published in 1615, proved by the German astronomer and mathematician Johannes Kepler (1571 - 1630), and later named Guldin's rules after the Swiss mathematician Paul Guldin (1577 - 1643), see Fig. 1a. The rules were known to Pappus of Alexandria, who stated them in his 12-volume work "Syna-goge", of which the first, second and last volumes have been lost. The Synagoge is considered one of the best sources of information on ancient Greek mathematics.

In this work we present newly constructed orthogonal curvilinear coordinate systems over various revolving surfaces (see Fig. 1b), as well as the nonlinear dynamics of rolling, without sliding, a heavy, homogeneous and isotropic ball on revolving surfaces (see Fig. 1c and d).



a* b* c* d*

Fig. 1. a) Paul Guldin (1577 - 1643); b) Coordinate surfaces and coordinate lines of a curvilinear orthogonal coordinate system constructed over a revolving biquadratic parabolic surface; kinetic parameters of rolling a heavy ball on a revolving parabolic; c) and biquadratic and d) parabolic surface.

2. Newly constructed orthogonal curvilinear coordinate systems over various revolving surfaces

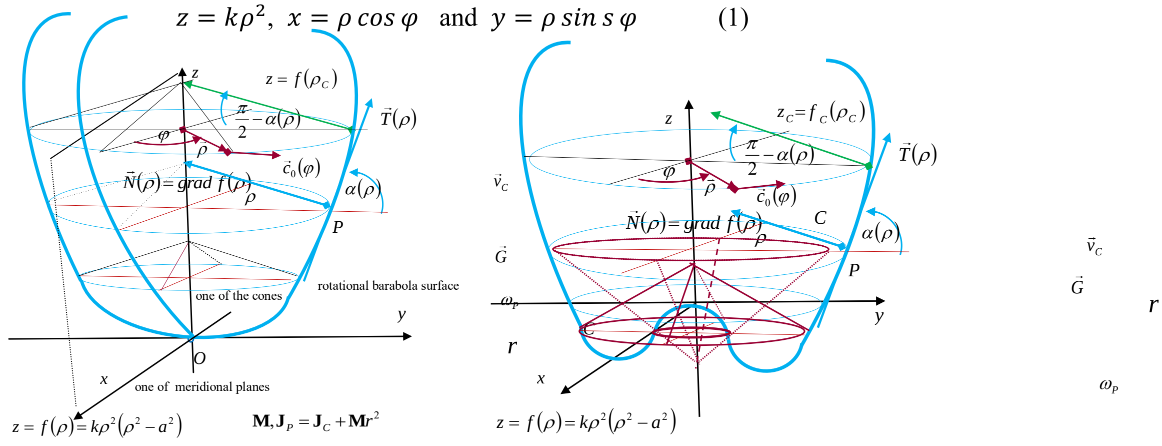
2.1 The geometry of the curvilinear coordinate system under a parabolic revolving surface

In this part, the geometry of the newly constructed curvilinear coordinate system under a parabolic revolving surface is presented by images, formulas and words, as well as by basic tangent vectors at points in intersection of the coordinate surfaces using curvilinear coordinates.

Fig. 2a presents the geometry of the curvilinear coordinate system under a parabolic revolving surface. The elements are the coordinate parabolic revolving surfaces and curvilinear coordinate lines of a parabolic revolving surface, system of cones and meridional planes, while the coordinate lines of the parabolic revolving surface are parabolas, straight lines and circles.

Parametric equations of the parabolic revolving surface, via curvilinear coordinates, ρ, φ and z, in the parabolic revolving cylindrical curvilinear system are of the form:

$$z = k\rho^2,\ x = \rho \cos\varphi \ \text{ and } \ y = \rho \sin s\, \varphi \qquad (1)$$



a* b*

Fig.2. The geometry of the curvilinear coordinate system under a revolving surface; a) Coordinate surfaces and coordinate lines of a parabolic revolving surface: system of cones and meridional planes; the coordinate lines of the parabolic revolving surface are parabolas, straight lines and circles; b) Coordinate surfaces and coordinate curvilinear lines of a revolving biquadratic parabolic surface are: system of cones and fringed cones, and meridional planes; the coordinate lines of a revolving biquadratic parabolic surface system are biquadratic parabolas, straight lines and circles and concentric circles

The unit vectors, in the tangential direction $\vec{T}$, in the normal direction $\vec{n}$ and in the circular direction $\vec{c}_0$ of the natural orthogonal trihedron, at the point of intersection of the coordinate surfaces (see Ref [3]) of the newly constructed orthogonal curvilinear coordinate system over the revolving parabolic surface, are:

$\vec{T} = \frac{1}{\sqrt{1+4k^2\rho^2}}\left(\vec{\rho}_0 + 2k\rho\vec{k}\right), \vec{n} = \frac{1}{\sqrt{1+4k^2\rho^2}}\left(-2k\rho\vec{\rho}_0 + \vec{k}\right)$ and $\vec{c}_0 = -sin\varphi\vec{\imath} + cos\,\varphi\,\vec{\jmath}$ (2)

where k is the parameter of the parabola, $\vec{\rho}_0$ is the unit vector in the radial direction: $\vec{\rho}_0 = (cos\,\varphi\,\vec{\imath} + sin\,\varphi\,\vec{\jmath})$, and $\vec{k}$ is the unit vector in the direction of the axis of axial symmetry of the parabolic revolving surface. $\vec{\rho}_0$ and $\vec{k}$ are orthogonal.

2.2 The geometry of the curvilinear coordinate system under a biquadratic parabolic revolving surface

In this part, the geometry of the newly constructed curvilinear coordinate system under a biquadratic parabolic revolving surface is presented by images, formulas and words, as well as by basic tangent vectors at the points of intersection of the coordinate surfaces using curvilinear coordinates.

Fig. 2 presents the geometry of the curvilinear coordinate system under a biquadratic parabolic revolving surface. The elements are the coordinate biquadratic parabolic revolving surfaces and curvilinear coordinate lines of a biquadratic parabolas, system of cones and meridional planes, while the curvilinear coordinate lines of the biquadratic parabolas in the revolving biquadratic surface are biquadratic parabolas, straight lines and circles as well as concentric circles.

Parametric equations of the biquadratic parabolic revolving surface, via curvilinear coordinates, ρ, φ and z, in the parabolic revolving cylindrical curvilinear system are of the form:

$$z = k\rho^2(\rho^2 - a^2),\;\; x = \rho\,cos\,\phi\varphi \;\; and \;\; y = \rho\,sin\,\varphi \qquad (3)$$

The unit vectors, in the tangential direction $\vec{T}$, in the normal direction $\vec{n}$ and in the circular direction $\vec{c}_0$ of the natural orthogonal trihedron, at the point of intersection of the coordinate surfaces (see Ref [3]) of the newly constructed orthogonal curvilinear coordinate system over the revolving biquadratic parabolic surface, are:

$$\vec{T} = \frac{1}{\sqrt{1+4k^2\rho^2\left(2\rho^2 - a^2\right)^2}}\left(\vec{\rho}_0 + 2\rho k\left(2\rho^2 - a^2\right)\vec{k}\right),$$

$$\vec{n} = \frac{1}{\sqrt{1+4k^2\rho^2\left(2\rho^2 - a^2\right)^2}}\left\langle -2k\rho\left(2\rho^2 - a^2\right)\vec{\rho}_0 + \vec{k}\right\rangle \text{ and } \vec{c}_0 = -sin\varphi\vec{\imath} + cos\,\varphi\,\vec{\jmath} \qquad (4)$$

where k and a^2 are the parameters of the biquadratic parabola, $\vec{\rho}_0$ is the unit vector in the radial direction: $\vec{\rho}_0 = (cos\,\varphi\,\vec{\imath} + sin\,\varphi\,\vec{\jmath})$, and $\vec{k}$ is the unit vector in the direction of the axis of axial symmetry of the parabolic revolving surface. $\vec{\rho}_0$ and $\vec{k}$ are orthogonal.

3. THE NONLINEAR DYNAMICS OF ROLLING, WITHOUT SLIDING, A HEAVY, HOMOGENEOUS AND ISOTROPIC BALL ON REVOLVING SURFACES

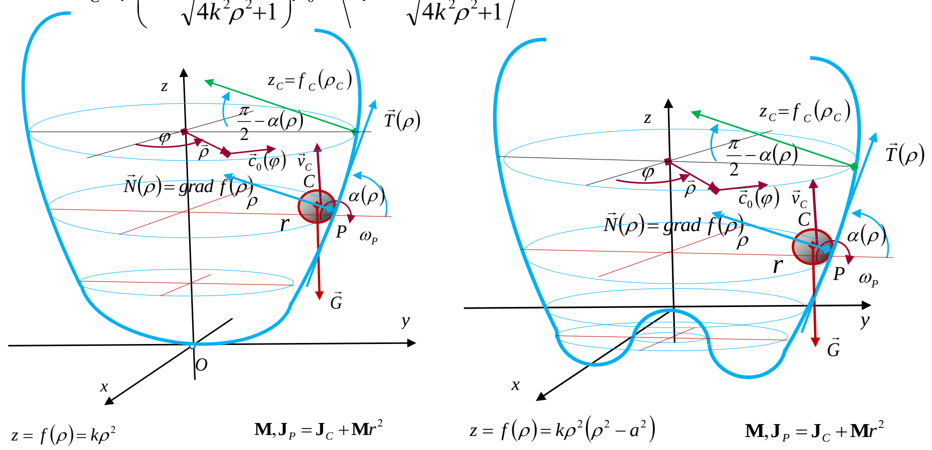
The dynamics of a ball rolling on a surface has holonomic stationary connections and two degrees of freedom of movement, and for the generalized coordinates we choose curvilinear orthogonal coordinates $\boldsymbol{\rho}$ and $\boldsymbol{\varphi}$.

3.1 Kinematic parameters of the nonlinear dynamics of rolling of a heavy, homogeneous and isotropic ball on a revolving parabolic surface

Fig. 3a* presents kinematic parameters of the nonlinear dynamics of rolling a heavy, homogeneous and isotropic ball on a revolving parabolic surface. The velocity of the center of mass of the ball and the point of contact of the rolling ball and the revolving parabolic surface are visible.

The position vector of the contact point, whose coordinates are $z = k\rho^2$, $x = \rho \cos\varphi$ and $y = \rho \sin\varphi$, is equal to: $\vec{r}_P = \rho(\cos\varphi\,\vec{\imath} + \sin\varphi\,\vec{\jmath} + k\rho\vec{k}) = \rho(\vec{\rho}_0 + k\rho\vec{k})$, while the position vector of the center of mass of a ball rolling on a revolving parabolic surface is $\vec{r}_C = \vec{r}_P + r\vec{n}$, where r is the radius of the ball.

$$\vec{r}_C = \rho\left(1 - \frac{2kr}{\sqrt{4k^2\rho^2+1}}\right)\vec{\rho}_0 + \left\langle k\rho^2 + \frac{r}{\sqrt{4k^2\rho^2+1}}\right\rangle\vec{k} \tag{5}$$



a* b*

Fig.3. Kinematic parameters of the nonlinear dynamics of rolling a heavy, homogeneous and isotropic ball a* on a revolving parabolic surface, and b* on a revolving biquadratic parabolic surface: the velocity of the center of mass of the ball at the point of contact of the rolling ball and the revolving surface

The velocity $\vec{v}_C$ of the center of mass C of the ball rolling, without sliding, on a revolving parabolic surface is the time derivative of its position vector $\vec{r}_C$ and is in the vector form:

$$\vec{v}_C = \dot{\rho}\left[1 - 2kr\left(4k^2\rho^2+1\right)^{-\frac{3}{2}}\right]\left(\vec{\rho}_0 + 2k\rho\vec{k}\right) + \dot{\varphi}\rho\left(1 - \frac{2kr}{\sqrt{4k^2\rho^2+1}}\right)\vec{c}_0 \tag{6}$$

From this expression (6), for the velocity $\vec{v}_C$ of the center of mass C of the ball in rolling, without sliding, on a revolving parabolic surface, we see that it has one component $\vec{v}_{C,c}$ in the circular direction orthogonal to the meridional plane through the axis of axis symmetry, as well as component $\vec{v}_{C,m}$ in two vector components in the meridional plane of the revolving parabolic surface:

$$\vec{v}_{C,c} = \dot{\varphi}\rho\left(1 - \frac{2kr}{\sqrt{4k^2\rho^2+1}}\right)\vec{c}_0 \text{ and } \vec{v}_{C,m} = \dot{\rho}\left[1 - 2kr(4k^2\rho^2+1)^{-\frac{3}{2}}\right]\left(\vec{\rho}_0 + 2k\rho\vec{k}\right) \quad (7)$$

Using these components $\vec{v}_{C,m}$ and $\vec{v}_{C,c}$, the velocity $\vec{v}_C$ of the ball's center of mass C in rolling, without sliding, on a revolving parabolic surface, as well as their squares, we can determine the orthogonal component of angular velocities $\vec{\omega}_{P,m}$ and $\vec{\omega}_{P,c}$ of ball's rolling, without sliding, along curvilinear coordinate lines on a revolving parabolic surface, with circles and derivative parabola lines:

$$\left[\vec{\omega}_{:,m}\right]^2 = \frac{\left[\vec{v}_{C,m}\right]^2}{r^2} = \frac{\dot{\rho}^2}{r^2}\left[1 - 2kr\left(4k^2\rho^2+1\right)^{-\frac{3}{2}}\right]^2\left(1+4k^2\rho^2\right)^2 \quad (8)$$

$$\left[\vec{\omega}_{P,c}\right]^2 = \frac{\left[\vec{v}_{C,c}\right]^2}{r^2} = \dot{\varphi}^2\frac{\rho^2}{r^2}\left(1 - \frac{2kr}{\sqrt{4k^2\rho^2+1}}\right)^2 \quad (9)$$

The expressions for the kinetic and potential energy, E_k and E_p, of a ball rolling, without slidding, on a revolving parabolic surface, as a function of two independent generalized coordinates, ρ and φ, are now (see Ref. [4]):

$$E_k(\rho,\dot{\varphi}) = \frac{1}{2}\mathbf{J}_P\left\{\frac{\dot{\rho}^2}{r^2}\left[1-2kr\left(4k^2\rho^2+1\right)^{-\frac{3}{2}}\right]^2\left(1+4k^2\rho^2\right)^2 + \dot{\varphi}^2\frac{\rho^2}{r^2}\left(1-\frac{2kr}{\sqrt{1+4k^2\rho^2}}\right)^2\right\} \quad (10)$$

$$E_p(\rho) = Mgh = Mgr\left(\frac{1}{\sqrt{1+4k^2\rho^2}} - \frac{1}{\sqrt{1+4k^2\rho_0^2 1}}\right) + Mgk\left(\rho^2 - \rho_0^2\right) \quad (11)$$

3.2 Kinematic parameters of the nonlinear dynamics of rolling a heavy, homogeneous and isotropic ball on a revolving parabolic surface

Two nonlinear differential equations of the nonlinear dynamics of rolling a heavy, homogeneous and isotropic ball on a revolving parabolic surface for independent generalized coordinates ρ and φ are obtained using Lagrange's differential equations of the second kind. If we introduce the rolling coefficient κ in the form (see Ref.[4]): $\kappa = \frac{J_P}{Mr^2} = \frac{i_P^2}{r^2} = \frac{i_C^2}{r^2} + 1$, where J_P is the axial mass inertia moment of the ball for the axis of rolling, M and r the mass and radius of the ball, then the derived system of two nonlinear differential equations of a rolling, no-slip, heavy homogeneous and isotropic ball on a revolving parabolic surface take simple forms. From the second nonlinear differential equation in the circular coordinate φ, the first integral is obtained in the following form:

$$\left\langle \dot{\varphi}\frac{\rho^2}{r^2}\left(1-\frac{2kr}{\sqrt{4k^2\rho^2+1}}\right)^2\right\rangle = C = const \quad \text{or in form} \quad \dot{\varphi} = C\left(\tfrac{r}{\rho}\right)^2\left(\frac{\sqrt{4k^2\rho^2+1}}{\sqrt{4k^2\rho^2+1}-2kr}\right)^2 \tag{12}$$

It is a cyclic integral (12), and the circular coordinate φ is a cyclic coordinate. Using that integral, which was introduced into the first nonlinear differential equation, we eliminate the cyclic circular coordinate φ from it, so that we get a nonlinear differential equation with only one independent generalized coordinate ρ in the following form:

$$\frac{\ddot{\rho}}{r^2}\left[1-2kr\left(4k^2\rho^2+1\right)^{-\frac{3}{2}}\right]^2\left(1+4k^2\rho^2\right)^2+\frac{1}{2}\frac{\dot{\rho}^2}{r^2}\frac{\partial}{\partial\rho}\left\langle\left[1-2kr\left(4k^2\rho^2+1\right)^{-\frac{3}{2}}\right]^2\left(1+4k^2\rho^2\right)^2\right\rangle-$$
$$-\frac{1}{2}C^2\left(\tfrac{r}{\rho}\right)^4\left(\frac{\sqrt{4k^2\rho^2+1}}{\sqrt{4k^2\rho^2+1}-2kr}\right)^4\frac{\partial}{\partial\rho}\left\{\frac{\rho^2}{r^2}\left(1-\frac{2kr}{\sqrt{4k^2\rho^2+1}}\right)^2\right\}+\frac{\partial}{\partial\rho}\left(\frac{g}{\kappa r}\frac{1}{\sqrt{4k^2\rho^2+1}}\right)+2\frac{g}{\kappa r}\rho=0 \tag{13}$$

Expressions for the kinetic and potential energy, (10) and (11), of a ball rolling, without slipping, on a revolving parabolic surface, as a function of two independent generalized coordinates, φ and ρ, are now in the form $E(\rho,\dot{\varphi})=E_k(\rho,\dot{\varphi})+E_p(\rho)=const=E(\rho_0,\dot{\varphi}_0)$. Then we use the cyclic integral (12), which we enter in the expression for the energy integral, and using it, we eliminate the derivative of the cyclic coordinate $\dot{\varphi}$ from the energy integral, so that we get the nonlinear equation, which represents the equation of the phase trajectory of the rolling, no-slip, heavy ball on a revolving parabolic surface. We can write that equation in the form:

$$\left\{\frac{\dot{\rho}^2}{r^2}\left[1-2kr\left(4k^2\rho^2+1\right)^{-\frac{3}{2}}\right]^2\left(1+4k^2\rho^2\right)^2\right\}+\left\{\frac{\rho^2}{r^2}C^2\left(\tfrac{r}{\rho}\right)^4\left(\frac{\sqrt{4k^2\rho^2+1}}{\sqrt{4k^2\rho^2+1}-2kr}\right)^4\left(1-\frac{2kr}{\sqrt{4k^2\rho^2+1}}\right)^2\right\}+$$
$$+\frac{g}{\kappa r}\left(\frac{1}{\sqrt{4k^2\rho^2+1}}-\frac{1}{\sqrt{4k^2\rho_0^2+1}}\right)+\frac{g}{\kappa r^2}k\left(\rho^2-\rho_0^2\right)=$$
$$=\left\{\frac{\dot{\rho}_0^2}{r^2}\left[1-2kr\left(4k^2\rho_0^2+1\right)^{-\frac{3}{2}}\right]^2\left(1+4k^2\rho_0^2\right)^2\right\}+\left\{\frac{\dot{\rho}_0^2}{r^2}C^2\left(\tfrac{r}{\rho_0}\right)^4\left(\frac{\sqrt{4k^2\rho_0^2+1}}{\sqrt{4k^2\rho_0^2+1}-2kr}\right)^4\left(1-\frac{2kr}{\sqrt{4k^2\rho_0^2+1}}\right)^2\right\} \tag{14}$$

By changing the initial positions ρ_0 and φ_0, i.e. the generalized phase coordinate ρ_0 and its time derivative $\dot{\rho}_0$ at the contact point of the ball and the revolving parabolic surface, at the initial moment of rolling, without slipping, we draw different phase trajectories by equation (13) in the phase plane $(\rho,\dot{\rho})$, on whose axes the generalized independent coordinate ρ and its derivative $\dot{\rho}$ are positioned, thus creating a phase portrait of nonlinear dynamics of the rolling, no-slip, heavy ball on a revolving parabolic surface.

3.3 Kinematic parameters of the nonlinear dynamics of the rolling of a heavy, homogeneous and isotropic ball on a revolving biquadratic parabolic surface

Fig. 3b* presents kinematic parameters of the nonlinear dynamics of rolling a heavy, homogeneous and isotropic ball on a revolving biquadratic parabolic surface. The velocity of the center of mass of the ball and the point of contact of the rolling ball and the revolving biquadratic parabolic surface are visible.

The velocity $\vec{v}_C$ of the center of mass C of the ball rolling, without sliding, on a revolving biquadratic parabolic surface is the time derivative of its position vector $\vec{r}_C$ and is in the vector form:

$$\vec{v}_C = \frac{d\vec{r}_C}{dt} = \dot{\rho}\left\langle \vec{\rho}_0 + 2k\rho\left(2\rho^2 - a^2\right)\vec{k}\right\rangle - +2kr\dot{\rho}\left(6\rho^2 - a^2\right)\left\langle \vec{\rho}_0 + 2k\rho\left(2\rho^2 - a^2\right)\vec{k}\right\rangle\left[1 + 4k^2\rho^2\left(2\rho^2 - a^2\right)^2\right]^{-\frac{3}{2}} + \\ + \rho\dot{\varphi}\left\langle 1 - \frac{2kr\left(2\rho^2 - a^2\right)}{\sqrt{1 + 4k^2\rho^2\left(2\rho^2 - a^2\right)^2}}\right\rangle\vec{c}_0 \tag{15}$$

From this expression (16) for the velocity $\vec{v}_C$ of the center of mass C of the ball, in rolling, without sliding, on a revolving biquadratic parabolic surface, we see that it has one component $\vec{v}_{C,c}$ in the circular direction orthogonal to the meridional plane through the axis of axis symmetry, as well as component $\vec{v}_{C,m}$ in two vector components in the meridional plane of the revolving biquadratic parabolic surface:

For this case of rolling a heavy ball on a biquadratic revolving surface, analogously to the contents of chapters 3.1. and 3.2, we can derive a system of two nonlinear differential equations, show that there is a cyclic integral as well as a cyclic coordinate, and then derive the energy integral and the phase trajectory equation. Generalizations for rolling a heavy ball on any revolving surface are also possible.

4. THEOREMS ON THE PROPERTIES OF THE NONLINEAR DYNAMICS OF ROLLING A HEAVY BALL ON REVOLVING SURFACES

On the basis of the results presented in the previous chapters, we can formulate a series of 15 theorems: five theorems about the constructed orthogonal curvilinear coordinate system on the revolving surface and ten theorems about the dynamics of rolling, without sliding, a heavy homogeneous and isotropic material ball on a revolving surface (with at least one axis of symmetry - sphere, cone, torus, revolving parabolic and biquadratic parabolic spheres, as well as general revolving surfaces - coordinate surfaces of curvilinear orthogonal systems of curvilinear coordinates).

Here are several conclusions, which follow from the contents of the previous chapters, and on the basis of which it is easy to formulate all 15 theorems:

1* two degrees of freedom of the rolling ball, when the ball's self-rotation is neglected;

2* one cyclic coordinate in circular direction and one cyclic integral;

3* the velocity of the center of mass of the rolling ball has two components: one in the meridional plane parallel to the tangent to the parabolic/biquadratic parabolic/general derivative line of the revolving parabolic/biquadratic parabolic/general surface on which the ball rolls, without slipping, as well as the second component in the circular direction, which is in the direction of the tangent to the circle - the curvilinear coordinate line of the revolving parabolic/biquadratic parabolic/general surface;

4* two components of the angular velocity of ball rolling, without slipping: 1. angular velocity of rolling without sliding, along the parabolic/biquadratic parabolic/general derivative of the revolving surface, and around the current axis of rolling through the point of current contact between the ball and the revolving surface, perpendicular to the meridional plane, and tangential in the circular direction on the circular coordinate line of the revolving surface and 2. angular velocity of rolling, without slipping, along the circular coordinate line of the revolving parabolic surface, and around the current axis of rolling through the point of instant contact between the ball and the revolving surface, perpendicular to the parabolic derivative of the revolving parabolic/biquadratic parabolic/general surface, and at an angle in relation to the plane of the coordinate circle of the revolving surface through the momentary contact point of the ball and the revolving surface - in the direction normal to that surface;

5* the instantaneous axes of the component rolling of the ball in the component rolling along the circular coordinate line lie on the coordinate conical surface;

6* the coordinate surfaces of the revolving parabolic/biquadratic parabolic/general surface on which the ball rolls, without slipping, are the revolving parabolic/biquadratic parabolic/general surface, the system of conical surfaces on which lie the normal to the base revolving surface and its meridional planes across its axis of symmetry. Coordinate lines are parabolas/biquadratic parabola/general lines, circles and cone derivatives - normal to the revolving surface;

7* energy integral;

8* two nonlinear differential equations of nonlinear dynamics of the rolling ball; It is possible to obtain a cyclic integral and one nonlinear differential equation expressed by only one independent generalized coordinate, which is not a cyclic coordinate.

9* a nonlinear equation of phase trajectories expressed by only one independent generalized coordinate, which is not a cyclic coordinate.

Due to the spatial limitations, we will only list one set of formulated theories.

Theorem 1. Fringed (Bounded) cones as coordinate surfaces of an orthogonal curvilinear coordinate system of a revolving surface occur when the revolving surface is formed by rotating a curved line with two or more minima in pairs. If the revolving surface has two minima, two series of coordinate surfaces appear as coordinate surfaces, one series in the form of cones and the other in the form of fringed cones. If the revolving surface has four minima, three series of coordinate surfaces, one in the form of a cone and two series of fringed cones, appear as coordinate surfaces.

Theorem 2. The velocity $\vec{v}_C$ of the center of mass C of a rolling ball, without slipping, has two components $\vec{v}_{C,m}$ and $\vec{v}_{C,c}$: one $\vec{v}_{C,m}$ in the meridional plane, parallel to the tangent $\vec{T}$ to the derivative of the revolving parabolic (or general revolving) surface on which the ball rolls, and the other component $\vec{v}_{C,c}$ in the circular direction $\vec{c}_0 = -\sin\varphi\,\vec{\imath} + \cos\varphi\,\vec{\jmath}$, which is in the direction tangent to the circle - the curvilinear coordinate line of the revolving parabolic (or general revolving) surface.

Theorem 3. Using the components $\vec{v}_{C,m}$ and $\vec{v}_{C,c}$, velocity $\vec{v}_C$ of the center of mass C of the ball rolling, non-sliding, on a revolving parabolic (or general revolving) surface as well as their squares, we can determine the component angular velocities $\vec{\omega}_{P,m}$ and $\vec{\omega}_{P,c}$ of component rolling, non-sliding, balls on curvilinear coordinate lines on a revolving parabolic (or general revolving) surface: circles $z = const$ and parabolas or corresponding

curvilinear coordinate $\varphi = const$ to derivatives of revolving parabolic (or general revolving) surface. **8.1 *** Angular velocity $\vec{\omega}_{P,m}$ of the component rolling along the coordinate line of the parabola - derivatives of the revolving parabolic surface, represented by a vector perpendicular to the meridional plane, with the current axis of rolling in the circular direction $\vec{c}_0 = -sin\varphi\vec{i} + cos\,\varphi\,\vec{j}$, orthogonally directed to the meridional plane. ***8. 2 **** The component angular velocity $\vec{\omega}_{P,c}$ of the roll, along a circular coordinate line, represented as a vector, is in the direction of the tangent to the corresponding parabolic coordinate line. The roll component, with the current roll axis, is in the direction of the normal determined by the unit vector $\vec{n} = \frac{1}{\sqrt{4k^2\rho^2+1}}\left(-2k\rho\vec{\rho}_0 + \vec{k}\right)$. This normal is perpendicular to the coordinate revolving surface of revolution and is not in the planes of the corresponding coordinate circle lines. It lies in the meridional plane at an angle $\alpha(\rho)$ with respect to the plane of the corresponding coordinate circle line and is always in the direction of one of the derivatives of the coordinate cones. The tangent of the angle $\alpha(\rho)$ of inclination with respect to the surface of the corresponding coordinate circle is given by $tg\alpha(\rho) = z' = f'(\rho) = 2k\rho$.
In the case of such chosen independent generalized coordinates φ and ρ, introducing the expression for the derivative of the cyclic coordinate from the cyclic integral into the nonlinear differential equation obtained by the independent generalized coordinate, we obtain a nonlinear differential equation in the coordinate ρ, which contains only the coordinate ρ.

5. Concluding Remarks

The series of ten theorems on the rolling, without sliding, of a heavy, homogeneous, isotropic ball on revolving surfaces, using a mathematical analogy [5], can also be applied to the movements of a heavy material point on revolving surfaces, where it is previously introduced that the radius of the ball (material particle) is equal to zero.

When selected, in both systems, the circular coordinate and the radial coordinate of the qualitative analogy (see Ref. [5]) proceed between the following kinetic parameters: in both systems, there are two degrees of freedom of movement, i.e. rolling by neglecting one's own rotation; a circular coordinate is a cyclic coordinate and it has a cyclic integral by that coordinate; in both dynamics the nonlinear differential equation in the radial coordinate is of the same type. The equations of the phase trajectories in both dynamics are analogous.

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