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NEW RHEOLOGIC MODELS OF THE FRACTIONAL TYPE

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Abstract. *This work presents seven novel fractional-type complex rheological models, each defined by specific structural formulas and fractional-order constitutive relations. These models incorporate fractional differential operators to describe the behavior of ideal materials. Two fundamental models are highlighted, with graphical representations of their structural configurations and corresponding fractional-order constitutive equations for normal stress and axial dilation.*
The study also introduces the concepts of the compensated subsequential elasticity surface and the stress relaxation surface, both expressed as functions of time and the fractional differentiation exponent. A comprehensive overview of the models is provided, including their Laplace-transformed solutions, which characterize the evolution of normal stress or axial deformation in response to external stimuli.
These seven models offer a unified framework for describing the mechanical behavior of idealized fractional materials, encompassing both elasto-viscous solids and viscoelastic fluids.

Key words: *Seven new rheological complex models of ideal materials of the fractional type, Newton's ideally viscous fluid flow of the fractional type, Differential constitutive relation, fractional order, Internal degrees of freedom of movement*

1. INTRODUCTION

The author reviewed a large number of publications available by abstract and in their entirety, based on keywords: rheology, rheological models of materials, constitutive relations of fractional order, rheological dynamic systems of fractional type, which were published by many publishers of scientific literature. Based on that, the following conclusions emerged: a* the largest amount of content on rheological models of ideal materials refers to classical rheological models (see Figure 1.). The integral form of these

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models is discussed in a chapter of the university textbook [1] on the Theory of Elasticity, authored by a lecturer at the Faculty of Mechanical Engineering in Niš.

b* Most of the content with rheological models are with applications in construction, on concrete and rock materials, then in the textile dyeing industry with application to cotton and yarn. Cotton or a combination of wool and cotton; Among the more recent publications are those with content on biomaterials and the application of complex material models to biomaterials but based on classic rheological models of materials with some additions, but more of a descriptive character, than with more serious mathematical contributions.

Given that the journal editors and reviewers require an introduction that includes a list of published works related to the manuscript topic—often resulting in excessive citations that serve more as descriptions than as genuine scientific references, since they do not inspire new results, the author faced the challenging task of selecting a number of papers. These selected works primarily involve applications of classical rheological models supported by experiments and are fundamentally based on models already covered in the author's previously cited university textbook [1].

These are the following classic models: Kelvin-Voight's basic model of an elasto-viscous solid body, Maxwell's basic model of a visco-elastic fluid, Bigham's basic model of an elastoplastic body, or Birgam's complex model of a visco-elastic fluid body (see Reference [1] and Figure 1).

Before presenting descriptive citations of several notable and original works on classical rheological models and their applications, including experimental approaches discussed therein, it is important to emphasize the author's new results. These results pertain to seven novel fractional-type complex rheological models, with applications in the dynamics of fractional-type rheological systems, such as oscillators and crawlers. There are no previously published results on this topic, the findings presented here are entirely new and original. They are exclusively inspired by the models discussed in the chapter of the already cited Reference [1], drawn from the author's work in the Theory of Elasticity.

Rheology (from Greek ῥέω rhéō, "flow" and -λογία, -logia, "study"), in short description, is the area of science of the flow of material, primarily in the liquid state, but also "soft solids" or solids under conditions where they react by plastic flow-yield, rather than elastically deforming in response to the applied force. Rheology is the science of deformation and flow within materials. It is a branch of physics that deals with the deformation and flow of materials, solids and liquids. The term rheology was coined by Eugene K. Bingham, a professor at Lafayette College, in 1920 at the suggestion of a colleague, Marcus Rayner.

In the paper [2], rheological characterization and rheological models for describing viscoelastic behavior of viscoelastic material were investigated. The classic linear models of Kelvin and Maxwell were used in parallel, i.e. in series, of Hooke's ideal elastic model and Newton's ideal viscous fluid. Also, see Ref. [3].

In the paper [4], the classic linear rheological basic models Kelvin-Voigt and Maxwell's model were used, as well as the modified Burgers model, which consists of the first two basic models applied to concrete in order to test concrete behavior. Here is indicated need for testing of the concrete and their characteristics in basic rheological models for service loads.

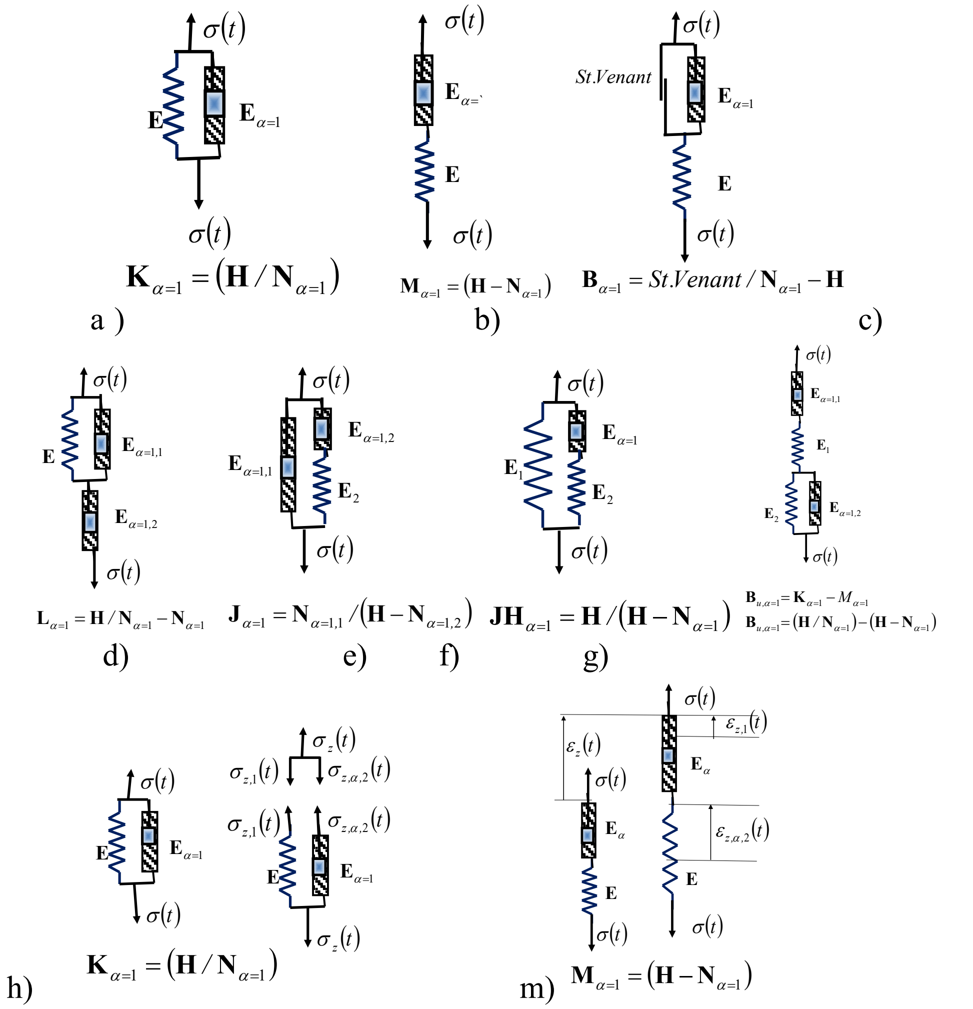


Figure 1. Classical complex rheological models of ideal materials with linear Newtonian model of viscous fluid: a) Basic complex Kelvin-Voigt model of elasto-viscous solid material; b) Basic complex Maxwell's model of viscoelastic fluid; c) Complex Bingham's model of elasto-visco-plastic solid material; d) Complex Lethersich's model of visco-elastic fluid; e) Complex Jeffrey's visco-elastic fluid model; f) Complex Jeffrey-H's model of an elasto-viscous solid body; d) Complex Burgers model of visco-elastic fluid; h) Decomposition and analysis of the state of normal stresses in the cross-section points of the basic complex Kelvin-Voight model of elasto-viscous solid material; m) Decomposition and analysis of the state of axial dilatations of the basic complex Maxwell model of a viscoelastic fluid (See Ref [1])

In Reference [5] in order to formulate the rheological model of time dependent deformations of soft rocks, laboratory tests on marl creep have been carried out. Basic task in mathematical description of time dependent deformations of a certain material is to define deformations as a function of time, stress and temperature. Experimental research on marl creep has been carried out. Wallner's rheological models, based on the results of salt-rock tests, approximate the marl creep behavior well. By the comparative analysis of time-deformations after total or partial unloading, it was concluded that the cited model does not include reversible deformation of creep, as the result of the assumption that primary creep depends only on current stress state. As suggested by the author of the paper [5], a modified rheological model, describing soft rock behavior after loading and total or partial unloading, is formulated. Primary creep in the model is formulated as a function of the preceding stress change that triggered the onset of creep.

In References [6,7], an overview of the basic classic simple and complex rheological models is given, and in particular, the complex classic Burgers model is used to describe the dilation of yarns of different compositions. The authors set up Burgers` classic wool, cotton and cotton/wool blend skeins. The results of the corresponding experiments carried out are also presented.

The study of mechanical properties of biological materials mathematical models of viscoelastic and viscoplastic properties of polymers, suspensions and gels might be very useful [8]. Implementation of fractional derivatives in modelling viscoelastic and plastic properties of materials is new trend in science [9, 10]. The general fractional-order Voigt and Maxwell models are used to describe rheological phenomena of real materials with the memory effect in [11].

References [13, 14] apply a fractional flow model to describe a time-dependent behavior of non-Newtonian substances. Specifically, author models the physical mechanism underlying the thixotropic and anti-thixotropic phenomena of non-Newtonian flow. This study investigates the behaviors of cellulose suspensions and starch-milk-sugar pastes under constant shear rate. The results imply that the presented model with only two parameters is adequate to fit experimental data. Moreover, the parameter of fractional order is an appropriate index to characterize the state of given substances. Its value indicates the extent of thixotropy and anti-thixotropy with positive and negative order, respectively.

Fractional rheology-informed neural networks for data-driven identification of viscoelastic constitutive models are presented in Ref. [15]. In this paper the complexity of the models typically goes hand in hand with that of the observed behaviors and can quickly become prohibitive depending on the choice of materials and/or flow protocols. Here, authors develop neural networks that are informed by a series of different fractional constitutive models.

Within the fractional derivative framework, paper [16] presents a study thermomechanical models with memory and compares them with classical Volterra theory. Fractional models involve significant differences in the type of kernels and predict important changes in the behavior of fluids and solids. An analogous analysis is carried out for the phenomenon of heat propagation with memory and presented in the paper.

Complete and integral theory of Analytical Dynamics (Mechanics) of Discrete Hereditary Systems are presented in Ref. [17] as well as some experiment to determine kernel of rheology for different rheological materials hereditary property.

Models of ideal materials: Functional dependencies of the state of stress and the state of deformation are called constitutive relations, which are determined by the internal physical properties of the material body. On the basis of these constitutive relations, mathematical theory, as well as part of the continuum mechanics, is further based. The basic physical properties of body materials are elasticity, viscosity, and creepability. Thermal conductivity and piezoelectricity of deformable bodies are also included in presented models. There are other deformable materials, which have properties of coupled fields of other physical properties, which will not be discussed here (See Fig. 1).

Fractional type rheological models: The core idea is to replace the classical Newtonian viscous element, commonly used in traditional rheological models (in parallel or series configurations), with modified fractional-order Newtonian fluid elements. The behavior of these models is then defined under conditions of constant normal stress or constant axial strain. Subsequently, the focus shifts to analyzing rheological dynamic systems—either of the oscillator or crawler-creep type—with a single degree of freedom (oscillation or creep). The goal is to determine the natural or forced response amplitudes as a function of the fractional differentiation exponent, varying within the interval from zero to one (see Figs 2 and 3).

2. NEW FRACTIONAL TYPE RHEOLOGIC MODELS OF IDEAL MATERIALS

Let us identify the features and properties inherent to each material body. The dominance of specific characteristics depends on both the internal structure of the material and external influences. These influences may include factors such as the magnitude of the applied load, the amount of transferred heat, the rate of loading, or the speed of deformation. Under the influence of an external load, a Newtonian (ideally viscous) fluid flows, as it offers no resistance to sheer stress. Consequently, once the load is removed, the fluid does not return to its original configuration prior to loading, indicating a permanent deformation.

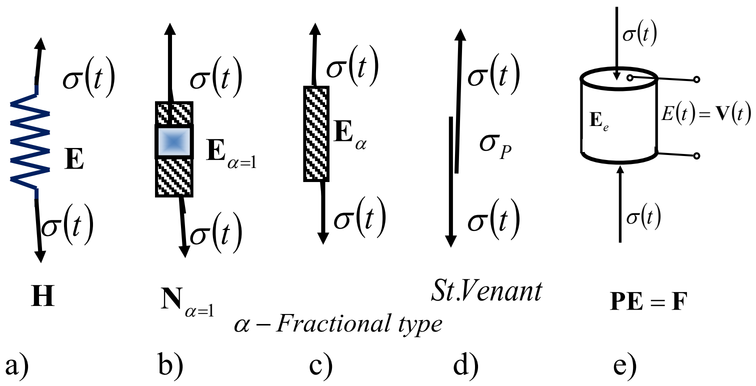


Figure 2. Elements of rheological models of ideal materials: a) Hooke ideal elastic element; b) Newton ideal viscous fluid linear element; c) New Newton ideal viscous fluid, fractional type, element; d) Bare de Saint Venant ideal plastic element; e) New Faraday ideal piezoelectric element (See Ref [1])

The yielding of a material of a solid body occurs when the intensity of the load exceeds the yield point for that material. Then we are talking about the plastic flow of material in a stressed body, as well as the accompanying plastic deformation. Plastic flow can take place only under mechanical load and can be lubricated so that it does not depend on time or temperature. After the end of the load, the material does not return to the configuration of the natural state before the deformation - the configuration of the undeformed body.

The difference between the properties of Newton's ideally viscous fluid, viscous flow, and Saint Venant's ideally plastic body, plastic flow, is that plastic flow requires the loading of the body to be such that the yield point is exceeded, while viscous flow can cause and quite a small load. In this case, after the cessation of the action of the load, the Saint Venant's ideally plastic body material does not return to the initial configuration of the natural state. The rate of deformation decreases as the load decreases and becomes equal to zero after the load is removed. When the loads are such that they do not exceed a certain yield point, the body is only deformed elastically.

The property of elasticity is reflected in the fact that after the cessation of the action of forces, the deformed Hooke's ideal elastic body returns to its natural state, without internal stresses and to the configuration of the original undeformed state.

In the previous part, we listed the basic mechanical properties of ideal bodies, the basic properties of the materials from which they are most often made. Each of these basic properties represents the basic classical rheological models of elasticity, viscosity and plasticity.

Real bodies have coupled enumerated properties, so more complex models with coupled structures can be considered.

The basic material models are (See Fig. 2.):

* Hooke's ideally elastic material – a solid body with the property of ideal elasticity (See Figure 2.a));

* Newton's ideal viscous material - viscous fluid with the property of viscous fluid flow (See Figure 2.b));

* Saint Venant's ideal plastic material - a rigid body with the property of plastic yielding when the load exceeds the yield point (See Fig. 2.d));

This manuscript is part of a comprehensive scientific research project focused on developing a new series of rheological models for materials, as well as discrete dynamic rheological systems of fractional type. The research has yielded multiple scientific contributions, with this manuscript representing the foundational work. It was initially submitted to the Journal *Applied Mathematical Modelling* one year ago. A month later, the journal informed the author of a high volume of pending submissions and proposed transferring the manuscript to one of four alternative Elsevier journals within the highest scientific category according to the KoBSON classification (M21a). The author selected the *Journal of Engineering Science* (M21a), where the manuscript remained under editorial consideration for nearly eight months before a second transfer proposal was made—again including *Applied Mathematical Modelling*, the original journal of submission. Meanwhile, four related articles presenting additional results from this research have been published in journals classified under the M21 category. The bibliographic details of these articles are listed in the reference section under entries [18–21], collectively representing the full scope of the new scientific findings.

In this paper, we intend to introduce two more ideal models based on their characteristics. The first new model is a generalization of the model of Newton's ideally viscous material based on the constitutive relationship between the normal stress and the rate of axial dilatation by derivation of non-integer order, by introducing a differential operator of non-integer, fractional order (Caputo derivative, see Refs 23-26]) $for \quad 0 <$

(for example, $\sigma_{z,\alpha} = \mathbf{E}_\alpha \mathsf{D}_t^\alpha[\varepsilon_z]$), in which is determined by an exponent $0 < \alpha < 1$ between zero and one. It is a generalization of the viscous dissipative element and includes it.

The second element we introduce is named after Faradey, and it is Faradey's piezo-electric element, which has the property of coupled fields of mechanical and electrical properties. The constitutive relation gives connections between mechanical normal stress σ_z and mechanical axial dilation ε_z with dielectric displacement and electric voltage of the electric field of polarization of the body, which is formed by the electric polarization of the material during the mechanical deformation and stress of the body.

The Fig. 2 shows the models of five of these five basic rheonomic elements.

We will illustrate the characteristic properties of ideal materials with elementary mechanical properties for the case of axial stress - an ideal homogeneous and isotropic state of normal stress in all points of the cross-section of a experimental resistance sample in the form of a rod or tube made of homogeneous and isotropic material stressed to axial stress. The figure shows the basic rheological elements, including the new basic model of fractional type, modified Newton's ideal viscous fluid material, fractional type, as well as the basic model of piezo-electric type Faraday's ideal piezo-electric material with coupled fields of mechanical and piezo- electrical state of the material.

We are now compiling a list of basic models of ideal materials, which we have already listed, according to their basic properties, and whose schematic representation is shown in the Fig. 2:

1* Model of Hooke's ideally elastic material - test tube axially stressed with indicated normal stress σ_z at the points of cross-sections and expansion-dilatation ε_z in the axial direction; The constitutive relation is $\sigma_z = \mathbf{E}\varepsilon_z$ and gives the connection between the normal stress σ_z in the cross-section of the test tube and the dilation $\varepsilon_z$of linear elements in the axial direction and the material constant $\boldsymbol{E}$, which is the modulus of elasticity and is determined experimentally. Basic dimensions: for normal stress σ_z is forces per unit area; for dilation ε_z, is the increment of length by length, or dimensionless size; for the modulus of elasticity $\boldsymbol{E}$ is of forces per square length, or surface area (See Fig. 2.a)).

2* Model of Newton's ideally viscous fluid – axial fluid flow stressed axially, with indicated normal stress σ_z at the points of transverse cross-sections of the fluid flow and velocity dilatation $\dot{\varepsilon}_z$ of linear fluid elements in the axial direction; The constitutive relation is $\sigma_z = \mathbf{E}_{\alpha=1}\dot{\varepsilon}_z$ and gives the connection between the normal stress σ_z in the fluid flow in the cross sections of the fluid flow and the dilation rate $\dot{\varepsilon}_z$ of linear elements in the axial direction of the fluid flow and the material constant $\boldsymbol{E}_{\alpha=1}$ of the fluid, which is the viscosity coefficient and is determined experimentally.

Basic dimensions: for normal stress σ_z is forces per unit area; for the rate of dilation $\dot{\varepsilon}_z$, the increase in length per total length, and in a unit of time, or a dimensionless quantity in a unit of time; for the viscosity coefficient $\boldsymbol{E}_{\alpha=1}$ is forces per square length, or surface area and time (See Fig. 2.b)).

3* Model of Bare de Saint-Venant's ideally plastic material - test tube stressed axially with indicated normal stress σ_z at the points of transverse sections; The normal stress σ_z is constant during plastic flow and does not depend on time and is equal to the plastic flow stress, which is determined experimentally for each material. The rate of dilation $\dot{\varepsilon}_z$ in the axial direction of linear elements during plastic flow is proportional to the flow stress, $\dot{\varepsilon}_z$ in which p material constant, which is determined experimentally for each material.

Basic dimensions: for normal stress σ_z is forces per unit area; for the rate of expansion $\dot{\varepsilon}_z$ of plastic flow, the increase in length per total length, and in a unit of time, or a dimensionless quantity in a unit of time; for the coefficient p of plastic flow, the square of the length, or the area per unit of force and unit of time (See Fig. 2.d)).

4* Model of generalized Newton 's ideally viscous fluid, fractional type - axial flow of fluid, fractional type, stressed axially, with indicated normal stress σ_z at the points of transverse transitions of the fluid flow and dilatation rate $\mathsf{D}_t^{\alpha}\left[\varepsilon_z\right]$, fractional type, of line elements of the fluid in the axial direction; The constitutive relation is $\sigma_z = \mathbf{E}_{\alpha}\mathsf{D}_t^{\alpha}\left[\varepsilon_z\right]$ and gives the connection between the normal stress σ_z in the axial flow of the fluid in the cross section of the fluid flow and the rate of dilation $\mathsf{D}_t^{\alpha}\left[\varepsilon_z\right]$, fractional type, of the linear elements of the fluid in the axial direction of the linear element flow of the fluid and the material constant $\boldsymbol{E}_{\alpha}$ of the fluid, fractional type, which is the viscosity coefficient of the fractional type fluid and is determined experimentally for each fluid, fractional type.

The structural mark is:

Basic dimensions: for normal stress σ_z, cross-sections of fluid flow, fractional type is forces per unit area; for the axial speed $\mathsf{D}_t^{\alpha}\left[\varepsilon_z\right]$ of dilatation of the fractional type of line elements of the axial direction of the fluid flow, fractional type, is length increment by total length, and in units of time depending on the order of fractional differentiation, or dimensionless quantity in units of times depending on the order of fractional differentiation; for the viscosity coefficient $\boldsymbol{E}_{\alpha}$ of forces per square of length, or surface area and times to a degree corresponding to the order of fractional differentiation (See Fig. 2.c)).

5* Model of Faraday's ideal piezoelectric material model $\boldsymbol{PE}$ - material with the property of polarization of transverse contour surfaces when the test tube is subjected to axial stress by compression or extension. Denote the normal stress σ_z at the points of the cross-sections by σ_z, the axial dilation by ε_z, and the electric voltage E of the polarization axis by E. The constitutive relation of Faraday's ideal piezoelectric material model are $E_z = -g\sigma_z$ and $\varepsilon_z = bE_z$, and gives the relationship between the mechanical state of the normal stress σ_z and the axial dilationε_z ε_z and the electric state of the electric voltage E of the polarization of the electric field of the piezoelectric material. Also, additional constitutive relations are in the forms: $D_z = b\sigma_z$ and $D_z = e\varepsilon_z$, which are the mechanical normal stress σ_z at the points of the cross sections, the line elements dilation of elements ε_z in the axial direction and the electric voltage

E of the electric field of polarization, and $D_z = b\sigma_z$ and $D_z = e\varepsilon_z$ are dielectric displacements, while where g, e and b are the constants of the piezoelectric material, mechanical and electrical. The units of the material constants of the piezoelectric ideal material are: $E\left[{}^{V}/_{m}\right]$, $g\left[{}^{Vm}/_{N}\right]$ and $b[m/V]$ (See Fig. 2.e)).

Such models of basic ideal materials with pure ideal properties can be combined into hybrid complex models, whereby one pair of models of basic materials can be connected in two ways (See Fig. 3.).

a* serial – in a series, which is indicated by a horizontal line "-" between the elements.

and

b* parallel, which is indicated by a vertical line "/" between the elements.

The picture shows the models of basic complex materials from two basic models of ideal materials, fractional type.

We highlight, now, especially two basic complex models of basic hybrid complex materials (See Figs 3. a) and b); h) and m)). Their structural formulas are composed of two basic elementary models of ideal materials: the model of Hook's ideal elastic material and the model of modified Newton's ideal viscous fluid, fractional type.

Connecting these two basic elementary models - elements is possible in parallel or in series. We will study the properties of these two basic hybrid complex material models, as they are found in each of the subsequent hybrid material models of hybrid more complex structures.

* Modified or generalized Kelvin's or Voigt's model, fractional type, which contains in its structure the basic model of Hooke's ideally elastic element and the basic model of Newton's ideally viscous element of fractional type connected in parallel (See Figs 3. a) and h)).

This model, a modified fractional-type Kelvin or Voigt model, is one of the two basic complex material models. It is one of the two basic complex models of ideal materials, of the fractional type, which contains their parallel connection and has no internal degrees of freedom of movement, but only one external degree of freedom of movement of one end of the element in relation to the other, for which we give detailed explanations in the section on rheological oscillators .

Modified Kelvin's or Voigt's model of the fractional type, denoted by and it is one of the two basic complex models of ideal materials, created from two basic models of ideal materials connected in parallel, Hooke's ideally elastic and modified Newton's fractional fluid type and has structural formula $\mathbf{K}_{\alpha}=\left(\mathbf{H}/\mathbf{N}_{\alpha}\right)$.

* Modified or generalized Maxwell's model of fractional type, is one of the two basic complex models of materials, $\boldsymbol{M}_{\alpha}$, ordinarily (serially) connected (See Figs 3. b) and m)) basic models of ideal materials Hooke's $\mathbf{H}$ ideally elastic and modified and Newton's ideal fluid $\boldsymbol{N}_{\alpha}$ fractional type has the structural formula $\boldsymbol{M}_{\alpha}=(\boldsymbol{H}-\boldsymbol{N}_{\alpha})$.

This means that the complex model contains in its structure serially connected basic models of Hooke's ideally elastic element and the basic model of Newton's ideally viscous element, fractional type.

This model, one of the two basic complex material models of the two basic models of identical materials, of the fractional type, which contains a series connection, has one internal degree of freedom of movement, in addition to one external degree of freedom of movement of one end of the complex model in relation to the other, for which we give detailed explanations in to the part about rheological dynamic systems of the crawler type.

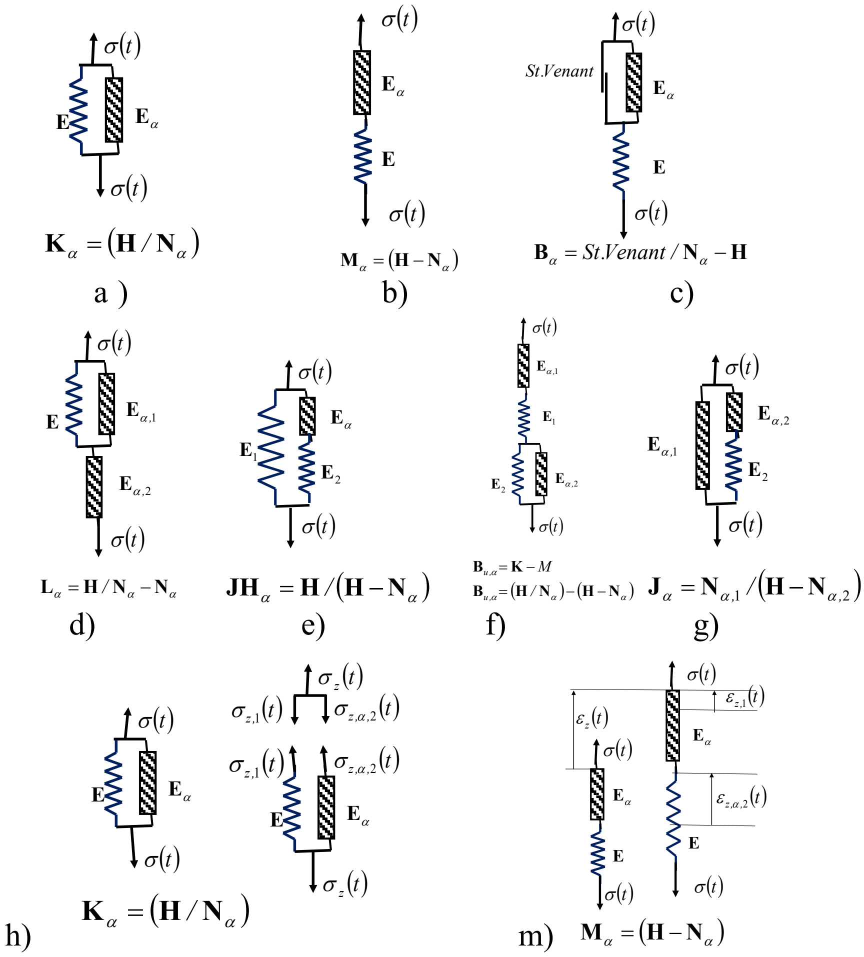


Figure 3. New generalized complex rheological models, fractional type, of ideal materials with generalized Newtonian element, fractional type, of viscous fluid: a) Generalized basic complex Kelvin-Voigt, fractional type, model of elasto-viscous solid material; b) Generalized complex Maxwell's, fractional type, model of viscoelastic fluid; c) Generalized complex Bingam's, fractional type, model of elasto-visco-plastic solid material; d) Generalized complex Lethersich's, fractional type, model of visco-elastic fluid; e) Generalized complex Jeffrey's, fractional type, visco-elastic fluid model; f) Generalized complex Jeffrey-H's, fractional type, model of an elasto-viscous solid body; d) Generalized complex Burgers, fractional type, model of visco-elastic fluid; h) Decomposition and analysis of the state of normal stresses in the cross-section points of the generalized complex Kelvin-Voigt, fractional type, model of elasto-viscous solid material; m) Decomposition and analysis of the state of axial dilatations of the generalized basic complex Maxwell, fractional type, model of a viscoelastic fluid

In both of the fundamental complex material models, the formation of the stress–strain relationship is significantly influenced by stress and dilatation, particularly through the rate of dilatation and the role of time.

Let us now examine in detail how time affects the relationship between stress and strain in these well-defined basic models of complex materials.

Now let's list the constitutive relations for simple elements - models based on individual properties of ideal materials:

* ideally elastic solid Hooke's element

$$\sigma_z = \mathbf{E}\varepsilon_z \tag{1}$$

* ideally viscous fluid Newtonian element

$$\sigma_z = \mu\dot{\varepsilon}_z \tag{2}$$

* ideally viscous fluid generalized Newton element of fractional type

$$\sigma_{z,\alpha} = \mathbf{E}_\alpha \mathsf{D}_t^\alpha[\varepsilon_z] \tag{3}$$

* ideally a piezoelectric Faraday element

$$E_z = -g\sigma_z$$
$$D_z = b\sigma_z \text{ and } D_z = e\varepsilon_z$$
$$\varepsilon_{ij} = b_{ijk}E_k, \tag{4}$$
$$D_i = b_{ijk}\sigma_{jk}(i,j,k = 1,2,3), \quad D_i = e_{ijk}\varepsilon_{jk}(i,j,k = 1,2,3),$$
$$\sigma_{ij} = -e_{kij}E_k, \varepsilon = bE.$$

3. Properties of New Fractional-Order Rheological Models for Ideal Materials and Their Constitutive Differential Relations

3.1 Modified or generalized Kelvin's or Voigt's model, now, fractional type surface.

A modified or generalized, fractional-type, Kelvin-Voigt's model, which is one of the two basic hybrid complex material models, and contains a parallel bound Hooke's ideally elastic element and a generalized Newtonian ideally viscous fluid element, fractional-type, whose constituents are relations between normal stress σ_z and axial dilatation ε_z, i.e. fast axial dilatation $\mathsf{D}_t^\alpha[\varepsilon_z]$, of the fractional type, in the form (see Fig. 4, decomposition idea):

$$\sigma_{z,1} = \mathbf{E}\varepsilon_z \tag{5}$$

$$\sigma_{z,\alpha.2} = \mathbf{E}_\alpha \mathsf{D}_t^\alpha[\varepsilon_z] \tag{6}$$

where $D_t^\alpha[\bullet]$ is the differential operator of fractional order is α , where the fractional order differentiation exponent α has values greater than zero and less than or equal to one: $0 < \alpha \leq 1$

In this paper, we will use the following differential operator of non-integer (fractional) order, defined by the following derivative and integral (Caputo fractional order derivative as differential operator fractional order α , see Refs [22, 25-30]):

$$\mathsf{D}_t^\alpha[\varepsilon_z(t)] = \frac{d^\alpha \varepsilon_z(t)}{dt^\alpha} = \varepsilon_z^{(\alpha)}(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{\varepsilon_z(\tau)}{(t-\tau)^\alpha}d\tau, \quad for \quad 0<\alpha\leq 1 \tag{7}$$

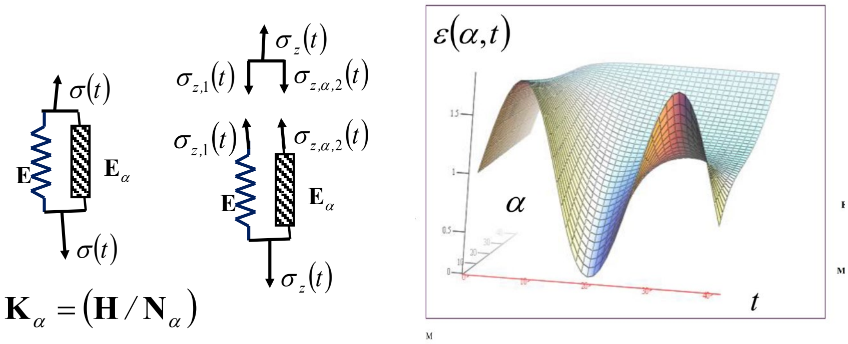
in the time interval in the branches from 0 to t, applied over the time function $\varepsilon_z(t)$. That is, applied to some independent generalized time-varying function, which in this case is axial dilatation $\varepsilon_z(t)$. In the previous differential operator $D_t^{\alpha}[\bullet]$ of fractional order, α is a rational number between 0 and 1, $0 < \alpha \leq 1$. α is a parameter-exponent of α^{th} fractional (non-integer) differentiation, fractional order α^{th} and α is determined experimentally, depending on the application for certain purposes. When the exponent is equal to zero, $\alpha = 0$, then the application of the differential operator, fractional order $D_t^{\alpha}[\bullet]$, gives the very same time function $\varepsilon_z(t)$- axial dilation, to which it was applied. When the exponent α, is equal to one, $\alpha = 1$, then the application of the differential operator, fractional order $D_t^{\alpha}[\bullet]$ gives the first-time derivative $\dot{\varepsilon}_z(t)$ of the time function to which it is applied. This property of the differential operator of fractional order $D_t^{\alpha}[\bullet]$, when its exponent α of fractional order, has values of rational numbers and in the interval $0 < \alpha \leq 1$, including its limit values, allows us to define with one expression both integer derivatives and derivatives of fractional order. And it is a useful mathematical description in various applications. In the previous definition of the differential operator of fractional order $D_t^{\alpha}[\bullet]$, the label represents the special Gamma function $\Gamma(1-\alpha)$, which is defined in the form of an integral (see Refs [9, 10, 12, 28]):

$$\Gamma(1-\alpha) = \int_0^{+\infty} e^{-t} t^{-\alpha} dt, \quad 1-\alpha > 0 \tag{8}$$

in the function of the exponent α of the differential operator of fractional order α, or in the general case in the function of the variable x, in the form:

$$\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1} dt, \quad x > 0 \quad \Gamma(x+1) = x\Gamma(x), \quad x > 0 \tag{9}$$

Generalized complex rheological models, fractional type, of ideal materials with generalized Newton element, fractional type, of viscous fluid: Generalized basic complex Kelvin-Voigt, fractional type, model of elasto-viscous solid material-decomposition and analysis of the state of normal stresses in the cross-section points of the generalized complex Kelvin-Voigt, fractional type, model of elasto-viscous solid material; b) surface of the subsequent elasticity of the modified Kelvin-Voigt model of the fractional type.



a) b)

Figure 4. The structure of a basic complex model of ideal materials, fractional type, and contributions of the properties in form of dilatation surfaces: a) structure of modified-generalized Kelvin-Voigt model of fractional type, which contains in its structure, parallel connected basic elements: Hooke's ideally elastic element and basic Newton's ideal viscous element of the fractional type; b) surface of the subsequent elasticity of the modified Kelvin-Voigt model of the fractional type in space coordinate: axial dilatation $\varepsilon_z(t)$, time t and exponent α of fractional derivative in interval from zero to one, $0 < \alpha \leq 1$, by expression (20)

The resulting normal stress at the ends of that modified Kelvin-Voigt model of the fractional type - the basic complex model of the ideal material, is equal to the sum of the normal stresses of the elements connected in parallel (see Fig. 4):

$$\sigma_z = \sigma_{z,} + \sigma_{z.\alpha,2} = \mathbf{E}\varepsilon_z + \mathbf{E}_\alpha \mathsf{D}_t^\alpha[\varepsilon_z] \tag{10}$$

In the case of rest of the modified Kelvin-Voigt model of the fractional type and at a very slow load change, when we can assume that the rate of dilation of the fractional type is small $\mathsf{D}_t^\alpha[\varepsilon_z] \to 0$ and tends to zero, the material behaves as the basic Hooke's ideally elastic material, and the normal stress of the material is almost proportional to dilation $\sigma_{z\backslash} \to \mathbf{E}\varepsilon_z$:

$$\mathsf{D}_t^\alpha[\varepsilon_z] \to 0 \qquad \Rightarrow \qquad \sigma_{z\backslash} \to \mathbf{E}\varepsilon_z \tag{11}$$

If the normal voltage at the ends of the modified Kelvin-Voigt model of fractional type suddenly increases from zero to some final value, $\sigma_{z,0} = const$, which remains constant in the following time interval, then we are surprised by the behavior of the model of this basic model of complex material.

If we assume that normal stress suddenly increases to some value $\sigma_{z,0} = const$and remains constant, then it is:

$$\mathbf{E}\varepsilon_z + \mathbf{E}_\alpha \mathsf{D}_t^\alpha[\varepsilon_z] = \sigma_{z,0} = const \tag{12}$$

In order to find the dependence of axial dilatation $\varepsilon_z(t)$ on time, at a constant value of the normal stress $\sigma_{z,0} = const$, which is exposed to the modified Kelvin-Voigt model, fractional type, it is necessary to solve the ordinary differential equation (12) of fractional order. To that end, we first apply the Laplace transform to the previous relation - an ordinary differential equation of fractional order, and obtain (see Refs [12, 28]):

$$\mathbf{E}\mathrm{L}\{\varepsilon_z\}+\mathbf{E}_\alpha \mathrm{L}\{\mathrm{D}_t^\alpha[\varepsilon_z]\}=\mathrm{L}\{\sigma_{z,0}\} \tag{13}$$

As: $\mathrm{L}\{1\}=\dfrac{1}{p}$, it follows that: $\mathrm{L}\{\varepsilon_z\}\langle \mathbf{E}+\mathbf{E}_\alpha p^\alpha\rangle=\dfrac{\sigma_{z,0}}{p}$, or in the form:

$$\mathrm{L}\{\varepsilon_z\}=\frac{\sigma_{z,0}}{p}\frac{1}{\langle \mathbf{E}+\mathbf{E}_\alpha p^\alpha\rangle} \tag{14}$$

The solution for the axial dilation $\varepsilon_z(t)$ as a function of time of the modified Kelvin-Voigt fractional-type model, the basic complex fractional-type model, when suddenly subjected to a constant normal stress $\sigma_{z,0}=const$ and held under that constant normal stress $\sigma_{z,0}$, is the inverse Laplace transform $\varepsilon_z(t)=\mathrm{L}^{-1}\mathrm{L}\{\varepsilon_z\}\backslash$ of the last expression (14):

$$\mathrm{L}\{\varepsilon_z\}=\frac{\sigma_{z,0}}{\mathbf{E}}\cdot\frac{1}{p}\cdot\frac{1}{\left\langle 1+\dfrac{\mathbf{E}_\alpha}{\mathbf{E}}p^\alpha\right\rangle} \tag{15}$$

Now, it is necessary to determine the approximate analytical expression as a function of time, $\varepsilon_z(t)=\mathrm{L}^{-1}\mathrm{L}\{\varepsilon_z\}\backslash$, as the inverse Laplace transform of the previous expression and cut in the time domain.

$$\varepsilon_z(t)=\mathrm{L}^{-1}\mathrm{L}\{\varepsilon_z\}=\mathrm{L}^{-1}\left\{\frac{\sigma_{z,0}}{\mathbf{E}}\cdot\frac{1}{p}\cdot\frac{1}{\left\langle 1+\dfrac{\mathbf{E}_\alpha}{\mathbf{E}}p^\alpha\right\rangle}\right\} \tag{16}$$

Therefore, the expression $\dfrac{\sigma_{z,0}}{\mathbf{E}}\cdot\dfrac{1}{p}\cdot\dfrac{1}{\left\langle 1+\dfrac{\mathbf{E}_\alpha}{\mathbf{E}}p^\alpha\right\rangle}$ must be developed in order of powers p, which is a complex number, using the formula (see Refs [26, 27]):

$$\frac{1}{1\pm x}=1\mp x+x^2\mp x^3+\cdots \tag{17}$$

And get the following line:

$$\mathrm{L}\{\varepsilon_z\}\approx\frac{\sigma_{z,0}}{\mathbf{E}}\cdot\left\langle\frac{1}{p}\cdot+\sum_{k=0}^{\infty}(-1)^k\left(\frac{\mathbf{E}_\alpha}{\mathbf{E}}\right)^k p^{k\alpha-1}\right\rangle \tag{18}$$

The inverse Laplace transform now gives an analytically approximate expression for the time-domain axial dilatation $\varepsilon_z(t)$ for the basic complex modified Kelvin-Voigt fractional-type model, when suddenly subjected to constant normal stress $\sigma_{z,0} = const$ and held under constant normal stress $\sigma_{z,0}$ in the following form:

$$\varepsilon_z(t) = \mathsf{L}^{-1}\mathsf{L}\{\varepsilon_z\} = \frac{\sigma_{z,0}}{\mathbf{E}} \cdot \mathsf{L}^{-1}\left\{\left\langle \frac{1}{p} + \sum_{k=0}^{\infty}(-1)^k \left(\frac{\mathbf{E}_\alpha}{\mathbf{E}}\right)^k p^{k\alpha-1}\right\rangle\right\} \tag{19}$$

The inverse Laplace transform of (19) gives $\varepsilon_z(t) = \mathsf{L}^{-1}\mathsf{L}\{\varepsilon_z\}\backslash$ and now gives an analytically approximate expression for the axial dilation $\varepsilon_z(t)$ in the time domain:

$$\varepsilon_z(t) = \frac{\sigma_{z,0}}{\mathbf{E}} \cdot \left\{1 + \sum_{k=0}^{\infty}(-1)^k \left(\frac{\mathbf{E}_\alpha}{\mathbf{E}}\right)^k \frac{t^{(2-\alpha)k+1}}{\Gamma(2k+2-\alpha k)}\right\} \tag{20}$$

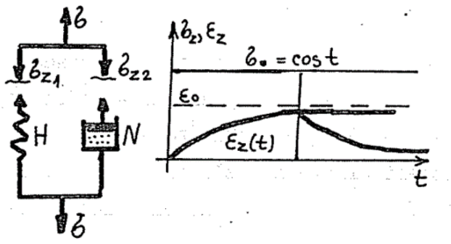
since it is:

$$\mathsf{L}^{-1}\left\{\frac{1}{p^2}\sum_{k=0}^{\infty}\frac{(-1)^k \omega_\alpha^{2k}}{p^{(2-\alpha)k+1}}\right\} = \sum_{k-0}^{\infty}(-1)^k \omega_\alpha^{2k} \frac{t^{(2-\alpha)k+1}}{\Gamma(2k+2-\alpha k)} = \mathsf{L}^{-1}\left\{\sum_{k=0}^{\infty}(-1)^k \omega_\alpha^{2k} p^{\alpha k-1}\right\}$$

In Fig. 2., the structure of a basic complex Kelvin-Voigt model of ideal materials, fractional type, and contributions of the properties in form of dilatation surfaces are presented. Fig. 2.a) shows the structure of modified-generalized Kelvin-Voigt model of fractional type, which contains in its structure, parallel connected basic elements: Hooke's ideally elastic element and basic Newton's ideal viscous element of the fractional type. Fig. 2.b) shows the surface of the *subsequent elasticity* of the modified Kelvin-Voigt model of the fractional type in space coordinate: axial dilatation $\varepsilon_z(t)$, time t and exponent α of fractional derivative in interval from zero to one, $0 < \alpha \le 1$.

Kelvin-Voigt classical model is special case of previous generalization**.** In the case when $\alpha = 1$, respectively $\boldsymbol{E}_{\alpha=1} = \mu$, when it is a classical Newton ideal viscous fluid in which the normal stress is proportional to the rate of axial dilation $\varepsilon_z(t)$, that is, to the first derivative of the dilation, then, if the Kelvin-Voigt classical model is suddenly subjected to the normal stress (see the diagram on the right in Fig. 4.b)) and keep the normal stress constant over time, the dilatation continues to increase with time, up to some value $\varepsilon_{z,0}$, for which it is $\varepsilon_z = \frac{\sigma_0}{E}\left(1 - e^{-\frac{E}{E_{\alpha=1}}t} +\right) \le \varepsilon_{z,0}$, and then begins to decrease with time to zero $\varepsilon_z = \frac{\sigma_0}{E} e^{-\frac{E}{E_{\alpha=1}}t}$.

Fig. 5 a) (in left) shows the decomposition of the Kelvin-Voigt classical model. Fig.5 b) (in right) shows graph of the axial dilatation, in the case that the Kelvin-Voigt classical model is suddenly subjected to the normal stress and keep the normal stress constant over time, the dilatation continues to increase with time, up to some value $\varepsilon_{z,0}$, $\varepsilon_z \le \varepsilon_{z,0}$, and then begins to decrease with time to zero by time function in the form $\varepsilon_z = \frac{\sigma_0}{E} e^{-\frac{E}{E_{\alpha=1}}t}$.



a) b)

Figure 5. The Figure (left) shows a) the decomposition of the Kelvin-Voigt classical model; b) if the Kelvin-Voigt classical model is suddenly subjected to the normal stress and keep the normal stress constant over time, the dilatation continues to increase with time, up to some value $\varepsilon_{z,0}$, and then begins to decrease with time to zero

If, on the other hand, the dilatation is limited to be constant over time, then the dilatation rate is equal to zero, and the normal stress (see the picture below sketch Figure 5.b)) does not decrease with time, the normal stress remains constant.

From the last analogy and the expression for the change of axial dilation, we see that the dilation of the classical Kelvin-Voigt model approaches asymptotically the limit value with the passage of time. This material behaves similarly during sudden unloading. Figure 5.b) (sketch on the right) shows the dependences of normal stress and axial dilatation on time. From that graph, one can see the characteristic property of viscoelasticity of the material, the lagging behind the change in axial dilatation after the normal stress. *This property is also subsequent elasticity*. Although in this material there is a viscous resistance to deformation, it is still a solid body, not a fluid. Voigt studied this material by studying the phenomenon of damping oscillations in crystals.

3.2 Modified or Generalized Fractional-Order Maxwell Model

Modified or generalized Maxwell's complex basic model, of fractional type, $\boldsymbol{M}_\alpha$, whose structural formula is $\boldsymbol{M}_\alpha = (\boldsymbol{H} - \boldsymbol{N}_\alpha)$, when Newton's viscous element of fractional type $\boldsymbol{N}_\alpha$ is entered into its structure, instead of classical Newton's viscous element $\boldsymbol{N}$, a regular connection with Hooke's ideally elastic element $\boldsymbol{H}$.

This order-series connection of Hooke's ideally elastic element $\boldsymbol{H}$ and Newton's viscous element $\boldsymbol{N}_\alpha$, of the fractional type, has the property that, throughout the entire basic complex and modified Maxwell model, of the fractional type $\boldsymbol{M}_\alpha$, the resulting rate of dilation of the fractional type $\mathsf{D}_t^\alpha[\varepsilon_z]$ is equal to the sum of the rates of dilation of the components of the order-in series connected elements, $\mathsf{D}_t^\alpha[\varepsilon_{z,1}]$ and $\mathsf{D}_t^\alpha[\varepsilon_{z,2}]$. In the sum of dilatation rates, of the fractional type, we use the dilation rates of the fractional type $\mathsf{D}_t^\alpha[\varepsilon_{z,1}]$ and $\mathsf{D}_t^\alpha[\varepsilon_{z,2}]$.

The normal stress σ_z at the points of the cross-section is the same throughout the entire modified Maxwell's model $\boldsymbol{M}_\alpha$, of the fractional type, so we can write the constitutive relations of each basic rheological element in this series connection, so the normal stresses σ_z in the dilatation function are (see Fig. 6.a):

$$\sigma_z = \mathbf{E}\varepsilon_{z,1} \tag{21}$$

$$\sigma_{z,\alpha} = \mathbf{E}_\alpha \mathsf{D}_t^\alpha[\varepsilon_{z.2}] = \sigma_z \tag{22}$$

The rate of dilation of the fractional-type modified Maxwell model $\mathsf{D}_t^\alpha[\varepsilon_z]$, the fractional-type equal to the sum of the dilation rates, the fractional-type Newtonian viscous element $\mathsf{D}_t^\alpha[\varepsilon_{z,1}]$ of the fractional-type and the Hooke's ideally elastic element $\mathsf{D}_t^\alpha[\varepsilon_{z,2}]$, in the form:

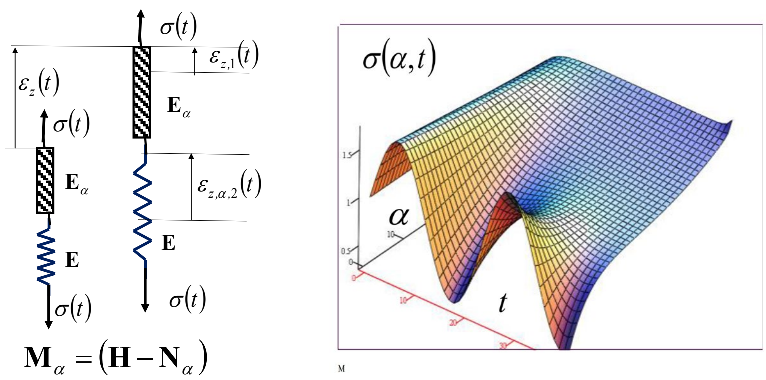
$$\mathsf{D}_t^\alpha[\varepsilon_z] = \mathsf{D}_t^\alpha[\varepsilon_{z,1}] + \mathsf{D}_t^\alpha[\varepsilon_{z,2}] \tag{23}$$

Since: $\varepsilon_{z,1} = \frac{\sigma_z}{E}$ it follows $\mathsf{D}_t^\alpha[\varepsilon_{z.1}] = \frac{1}{\mathbf{E}}\mathsf{D}_t^\alpha[\sigma_z]$ and $\mathsf{D}_t^\alpha[\varepsilon_{z.2}] = \frac{\sigma_z}{\mathbf{E}_\alpha}$, it follows that resalting dilatation is a sum (see Fig. 4 a) of component axial dilatation:

$$\mathsf{D}_t^\alpha[\varepsilon_z] = \frac{1}{\mathbf{E}}\mathsf{D}_t^\alpha[\sigma_z] + \frac{\sigma_z}{\mathbf{E}_\alpha} \tag{24}$$

When the fractional-type rate $\mathsf{D}_t^\alpha[\sigma_z]$ of change of normal stress approaches *to* zero $\mathsf{D}_t^\alpha[\sigma_z] \to 0$, the material described by the modified fractional Maxwell model behaves like a viscous fluid. This is because the deformation, i.e. axial dilation ε_z of the body: $\sigma_z \to \mathbf{E}_\alpha \mathsf{D}_t^\alpha[\varepsilon_z]$, increases indefinitely without any additional load. Upon unloading, the deformation in the ideally elastic (Hookean) element fully recovers, while the deformation resulting from the flow in the fractional-type viscous (modified Newtonian fluid) element $\mathbf{N}_\alpha$, connected in series, remains unrecovered.

If this material, a the fractional-type modified Maxwell model $\boldsymbol{M}_\alpha$ is suddenly loaded to some value of normal stress $\sigma_{z,0}$, the corresponding elastic deformation occurs instantaneously in Hooke's ideal elastic element $\varepsilon_{z,0} = \frac{\sigma_{z,0}}{E}$. This occures due to the sudden application of load at the very beginning of the observation period, the flow behavior of the serially connected fractional-type viscous element (modified Newtonian fluid) in the fractional Maxwell model does not immediately manifest. If the development of deformation (dilatation) is constrained—i.e., if the fractional-type rate of dilatation tends to zero $\mathsf{D}_t^\alpha[\varepsilon_z] \to 0$—then the normal stress becomes a time-dependent function that must be determined.



a) b)

Figure 6. The structure of second basic complex model of ideal materials and contributions of the properties in form of normal stress surface: a) Structure of the modified Maxwell model of the fractional type, which contains in its structure, connected in series: the basic Hooke's ideally elastic element and the basic Newton's ideally viscous element of the fractional type; b) Stress relaxation surface of the modified Maxwell model of the fractional type, in the coordinate system: normal stress σ_z, time t and exponent α of fractional derivative in interval from zero to one, $0 < \alpha \leq 1$, by expression (30)

When the rate of change of the normal stress $\mathsf{D}_t^\alpha[\sigma_z]$ of the the fractional-type modified Maxwell complex element $\boldsymbol{M}_\alpha$ tends to zero $\mathsf{D}_t^\alpha[\sigma_z]\to 0$, then the normal mechanical stress tends to a value proportional to the rate of dilation of the fractional type $\sigma_z \to \mathbf{E}_\alpha \mathsf{D}_t^\alpha[\varepsilon_z]$:

$$\mathsf{D}_t^\alpha[\sigma_z]\to 0 \Rightarrow \qquad \sigma_z \to \mathbf{E}_\alpha \mathsf{D}_t^\alpha[\varepsilon_z] \tag{25}$$

In order to determine the dependence of normal stress on time, when we keep the material model of the modified Maxwell complex model, fractional type $\boldsymbol{M}_\alpha$, at some constant rate of dilatation, fractional type $\left\{\mathsf{D}_t^\alpha[\varepsilon_z]_{z,,0}\right\} = const$, we write that:

$$\frac{1}{\mathbf{E}}\mathsf{D}_t^\alpha[\sigma_z] + \frac{\sigma_z}{\mathbf{E}_\alpha} = \left\{\mathsf{D}_t^\alpha[\varepsilon_z]_{z,,0}\right\} = const \tag{26}$$

By applying the previous condition (26), we obtain a differential equation of fractional order, which can be solved using the Laplace transform. We then apply the Laplace transform to the functional relationship defined by equation (26)—the fractional-order differential equation. As a result of this transformation, we obtain the following expression: $\frac{1}{\mathbf{E}}\mathsf{L}\left\{\mathsf{D}_t^\alpha[\sigma_z]\right\} + \frac{1}{\mathbf{E}_\alpha}\mathsf{L}\{\sigma_z\} = \mathsf{L}\left\{\mathsf{D}_t^\alpha[\varepsilon_z]_{z,,0}\right\}$. Then, by arranging the previous relation (26), we get:

$$\mathsf{L}\{\sigma_z\}\left\langle \frac{1}{\mathbf{E}_\alpha} + \frac{1}{\mathbf{E}}p^\alpha \right\rangle = \frac{1}{p}\left\{\mathsf{D}_t^\alpha[\varepsilon_z]_{z,,0}\right\} \tag{27}$$

That is, by solving the obtained relation by the Laplace transform $\mathcal{L}\{\sigma_z\}$, it follows that the Laplace transform of the normal stress $\mathcal{L}\{\sigma_z\}$, in the modified Maxwell complex element $\boldsymbol{M}_\alpha$, is of the fractional type in the form:

$$\mathsf{L}\{\sigma_z\} = \mathbf{E}_\alpha \left\{\mathsf{D}_t^\alpha [\varepsilon_z]_{z,,0}\right\} \cdot \frac{1}{p} \cdot \frac{1}{\left\langle 1 + \frac{\mathbf{E}_\alpha}{\mathbf{E}} p^\alpha \right\rangle} \tag{28}$$

Now it is necessary to determine an approximate analytical expression for the normal stress $\sigma_z(t)$ as a function of time, a modified fractional Maxwell complex element $\boldsymbol{M}_\alpha$ as the inverse Laplace transformation $\sigma_z(t) = \mathcal{L}^{-1}\mathcal{L}\{\sigma_z\}$ of the previous expression (28) and move from the complex domain to the time domain.

That's why we'll develop the expression $\mathbf{E}_\alpha \left\{\mathsf{D}_t^\alpha [\varepsilon_z]_{z,,0}\right\} \cdot \frac{1}{p} \cdot \frac{1}{\left\langle 1 + \frac{\mathbf{E}_\alpha}{\mathbf{E}} p^\alpha \right\rangle}$ in order of powers by p, which is a complex number, using the previously cited formula (17), and follows:

$$\mathsf{L}\{\sigma_z\} \approx \mathbf{E}_\alpha \left\{\mathsf{D}_t^\alpha [\varepsilon_z]_{z,,0}\right\} \cdot \left\langle \frac{1}{p} + \sum_{k=0}^{\infty} (-1)^k \left(\frac{\mathbf{E}_\alpha}{\mathbf{E}}\right)^k p^{k\alpha - 1} \right\rangle \tag{29}$$

The inverse Laplace transform $\sigma_z(t) = \mathcal{L}^{-1}\mathcal{L}\{\sigma_z\}$ of the previous expression$\mathcal{L}\{\sigma_z\}$, (29) , now gives an approximate analytical expression for the normal stress $\sigma_z(t)$ in the time domain, a modified fractional Maxwell complex element $\boldsymbol{M}_\alpha$, in power-order by time degrees form, of the form:

$$\sigma_z(t) = \mathsf{L}^{-1}\mathsf{L}\{\sigma_z\} \approx \mathbf{E}_\alpha \left\{\mathsf{D}_t^\alpha [\varepsilon_z]_{z,,0}\right\} \cdot \left\{ 1 + \sum_{k=0}^{\infty} (-1)^k \left(\frac{\mathbf{E}_\alpha}{\mathbf{E}}\right)^k \frac{t^{(2-\alpha)k+1}}{\Gamma(2k + 2 - \alpha k)} \right\} \tag{30}$$

because the:

$$\mathsf{L}^{-1}\left\{ \frac{1}{p} \sum_{k=0}^{\infty} \frac{(-1)^k \omega_\alpha^{2k}}{p^{(2-\alpha)k+1}} \right\} = \sum_{k-0}^{\infty} (-1)^k \omega_\alpha^{2k} \frac{t^{(2-\alpha)k}}{\Gamma(2k + 1 - \alpha k)}$$

Fig. 4 presents generalized complex rheological Maxwell models of fractional type for ideal materials, incorporating a generalized Newtonian viscous element of fractional order. In particular, Fig. 6.a illustrates the generalized fractional Maxwell model of a viscoelastic fluid, along with the decomposition and analysis of axial dilatation and normal stress states. The normal stress relaxation surface is shown, based on the approximate analytical expression (30), representing the time-dependent normal stress response of the fractional Maxwell model in a three-dimensional coordinate system: normal stress σ_z, time t and exponent α of fractional derivative in interval from zero to one, $0 < \alpha \leq 1$.

From the previous solution (30), as well as from the surface plot shown in Fig. 6b, it is evident that the normal stress $\sigma_z(t)$ decreases asymptotically over time and tends toward zero, as illustrated in Fig. 6.b. This gradual reduction in normal stress under constant dilatation is known as *normal stress relaxation* of fractional type.

The material under investigation-represented by the modified Maxwell complex element $\boldsymbol{M}_{\alpha}$ of fractional type-exhibits viscoelastic fluid behavior. This model is particularly suitable for describing the mechanical response of metals at very high temperatures, as well as the behavior of betaines.

Classical Maxwell's model with linear Newton element. In the case when $\alpha = 1$, respectively, $\boldsymbol{E}_{\alpha=1} = \mu$when dealing with a classical Newtonian ideal viscous fluid in which the normal stress at the points of the cross-section of the fluid flow is proportional to the velocity of axial dilation, $\sigma_z = \mu\dot{\varepsilon}_z$ i.e. the first derivative of axial dilation, then, if the classical Maxwell element, suddenly subjected to that normal stress (see the upper sketch in the Fig. 7.) and keeping the normal stress constant over time, the axial dilatation continues to increase with time ccording to the law $\varepsilon_z = \varepsilon_0 \left(1 + e^{\frac{\boldsymbol{E}}{\boldsymbol{E}_{\alpha=1}}t}\right)$.

If, on the other hand, the axial dilation of the classical Maxwell element is limited to be constant over time, then the rate of axial dilation is equal to zero, and the normal shape (see the lower sketch in the Fig. 5) decreases with time according to the exponential law $\sigma_z = \sigma_0 e^{-\frac{\boldsymbol{E}}{\boldsymbol{E}_{\alpha=1}}t}$.

When the rate of change of the normal stress at the points of the cross-section of the material of the classic Maxwell element tends to zero $\dot{\sigma}_z \to 0$, that model of the material behaves like a viscous fluid, because: $\sigma_z \to \boldsymbol{H}_{\alpha=1}\dot{\varepsilon}_z$, because the deformation, ε_z that is, the axial dilatation ε_z, of that material grows indefinitely without an increase in the load and the corresponding normal stress. When the material of the classical Maxwell element is unloaded, the axial deformation in the Hooke's ideally elastic element completely disappears, while the deformation due to fluid flow in the classical Newtonian viscous element-viscous fluid does not disappear in the serial connection.

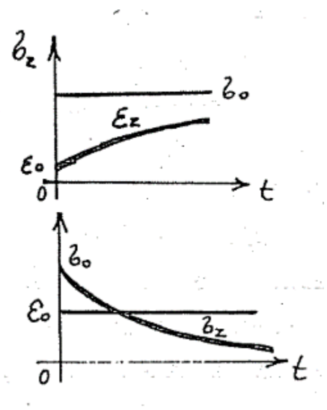


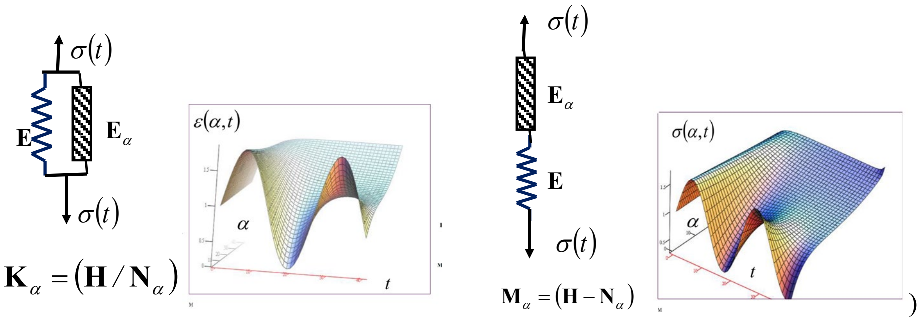
Figure 7 Graphs of stated of classical Maxwell model: of axial dilatation state during the time, in the case that normal stress is constant (upper graphs) and of normal stress during the time, in the case that axial dilatation is constant (lower graphs)

If this material, a classical Maxwell element, is suddenly loaded to some value of normal stress $\sigma_{z,0}$, it corresponds to an elastic axial deformation-dilation $\varepsilon_{z,0} = \frac{\sigma_{z,0}}{\boldsymbol{E}}$, created instantaneously in Hooke's ideally elastic element. This is because, due to the sudden load, immediately at the beginning of the observation of the behavior of the material of the classical Maxwell element, the flow, in the serially connected classical Newtonian viscous element-ideally viscous fluid, does not come to the fore. If we prevent the development of deformation - axial dilatation, assuming that the rate of dilatation tends to zero, $\dot{\varepsilon} \to 0$, then the normal stress is equal to: $\sigma_z = \sigma_0 e^{-\frac{E}{E\alpha=1}t}$.

As shown in Fig. 6 (b), the normal stress at points along the transverse cross-section decreases asymptotically over time and tends toward zero. This phenomenon—where normal stress decreases with time under constant axial dilatation—is known as *normal stress relaxation at cross-sectional points*. The material described by the classical Maxwell model behaves as a viscoelastic fluid and can be used to model the behavior of metals at very high temperatures, as well as the rheological properties of substances such as betaine.

3.3 Comparative Analysis of Generalized Fractional Rheological Models: Modified Kelvin–Voigt Elasto-Viscous Solid vs. Modified Maxwell Viscoelastic Fluid

The previous analysis of the features and properties of two basic complex rheologic material models, presented in Fig. 8. For comparison:



a) b) c) d)

Figure 8. The comparison of structure of two basic complex models of ideal materials and comparison of contributions of the properties in form of dilatation surfaces, i.e. normal stress surface: a) structure of modified Kelvin-Voigt model of fractional type, which contains in its structure, parallel connected basic elements: Hooke's ideally elastic element and basic Newton's ideal viscous element of the fractional type; b) surface of the subsequent elasticity of the modified Kelvin-Voigt model of the fractional type by expression (20); c) Structure of the modified Maxwell model of the fractional type, which contains in its structure, connected in series: the basic Hooke's ideally elastic element and the basic Newton's ideally viscous element of the fractional type; d) Stress relaxation surface of the modified Maxwell model of the fractional typem by expression (30);

*Modified Kelvin-Voigt model of the fractional type (Figs 8.a and b), which contains in its structure, parallel coupled, the basic model of Hooke's ideally elastic element and the basic model of the modified Newton's ideally viscous element, fractional type $\boldsymbol{N}_{\alpha}$, and that it is a solid body with the property of subsequent elasticity- It has a structural formula $\boldsymbol{K}_\alpha = (\boldsymbol{H}/\boldsymbol{N}_\alpha)$.

* Modified Maxwell model of fractional type (Figs 8.c and d) , is one of the two basic complex models of materials, sequentially (serially) connected basic elements of ideal materials, Hooke's ideally elastic and modified Newton's ideal viscous fluid, fractional type $\boldsymbol{N}_\alpha$, has the property of voltage relaxation. It has a structural formula $\boldsymbol{M}_\alpha = (\boldsymbol{H} - \boldsymbol{N}_\alpha)$.

To conclude, that two basic complex rheological models of materials, the modified Kelvin-Voigt model, fractional type $\boldsymbol{K}_\alpha$ and the modified Maxwell model, fractional type, $\boldsymbol{M}_\alpha$ composed of the same basic models of the elements of ideal materials, Hooke's ideally elastic and modified Newton's of an ideal viscous fluid, fractional type $\boldsymbol{N}_\alpha$, connected in parallel in , that is, in order (serial) in , exhibit two important and qualitatively different properties. The first model is a solid body with the property of **subsequent elasticity** $\boldsymbol{K}_\alpha = (\boldsymbol{H}/\boldsymbol{N}_\alpha)$ and the second model $\boldsymbol{M}_\alpha = (\boldsymbol{H} - \boldsymbol{N}_\alpha)$ is an elastoviscous fluid with the property of **normal stress relaxa**tion.

On the examples of the modified Kelvin-Voigt model of the fractional type $\boldsymbol{K}_\alpha$, as well as the modified Maxwell model of the fractional type $\boldsymbol{M}_\alpha$, we can see the structural formulas, which we defined based on the type of connection of the simple models, parallel or serial. These structural formulas are similar and in analogous with structural formulas in chemistry, as well as that the structure of rheological models can be expressed by rheological formulas. In the previous two examples, it can be seen that when connecting the basic elemental material models in series, the total elongation of the resulting rheological model is equal to the sum of the component elongations of the individual elements in the connection, while for the parallel joint the total normal stress is equal to the sum of the normal stresses, which occur in the parallel connected rheological elements.

4. STRUCTURES OF OTHER GENERALIZED MORE COMPLEX RHEOLOGICAL MODELS OF FRACTIONAL TYPE

The Figs 3. c), d), e), f) and g) show the structures of complex modified-generalized rheological models, fractional type:

* Fig. 3. c) shows generalized modified Bingam model of ideal complex material, fractional type, with rheological structural formula $\boldsymbol{B}_\alpha = St.Venant/\boldsymbol{N}_\alpha - \boldsymbol{H}$.

* Fig. 3. d) shows generalized modified Lethersich model of ideal complex material, fractional type, with rheological structural formula $\boldsymbol{L}_\alpha = \boldsymbol{H}/\boldsymbol{N}_\alpha - \boldsymbol{N}_\alpha$.

* Fig. 3. e) shows generalized modified Jeffry-H's model of ideal complex material, fractional type, with rheological structural formula $\boldsymbol{JH}_\alpha = \boldsymbol{H}/(\boldsymbol{H} - \boldsymbol{N}_\alpha)$.

* Fig. 3. f) shows generalized modified Jeffry's model of ideal complex material, fractional type, with rheological structural formula $\boldsymbol{J}_\alpha = \boldsymbol{N}_\alpha/(\boldsymbol{H} - \boldsymbol{N}_\alpha)$.

* Fig. 3. g) shows generalized modified Burgers model of ideal complex material of fractional type, with rheological structural fatmule $\boldsymbol{B}_{u,\alpha} = \boldsymbol{K} - \boldsymbol{M}$ or $\boldsymbol{B}_{u,\alpha} = (\boldsymbol{H}/\boldsymbol{N}_\alpha) - (\boldsymbol{H} - \boldsymbol{N}_\alpha)$.

There are other ideal complex materials, constructed to explain some mechanical phenomena or properties, let us only list a few:

* generalized modified Schwediff's model of ideal complex material of fractional type, with rheological structural formula $\boldsymbol{S}_{Schw,\alpha} = \boldsymbol{H} - (St.Venant/\boldsymbol{M}_\alpha)$.

* generalized modified Poyting or modified Thompson ideal complex material model, fractional type, with rheological structural formula $\boldsymbol{PTh}_{\alpha} = \boldsymbol{H}/\boldsymbol{M}_{\alpha}$.

* generalized modified Trouton's or modified Rankin's model, of ideal complex material, fractional type, with rheological structural formula $\mathbf{TR}_{\alpha}=\mathbf{N}_{\alpha}-\mathbf{PTh}_{\alpha}$.

* generalized modified Schofield-Scot-Vlair ideal complex material model, fractional type, with rheological structural formula $\mathbf{S}_{ch}\mathbf{S}_{c}\mathbf{B}_{\alpha}=\mathbf{S}_{chw,\alpha}-\mathbf{K}_{\alpha}$.

In all of these listed ideal complex models, it is possible to replace some of the simple material models with elementary Faraday's ideal piezo-electric model of the piezo-electric material, or to add that element to them, in series (order) or parallel connection and obtain new models of rheological piezo-electric models of ideal materials with specific properties that we program with a specific structure.

It is possible an arbitrary number of combinations of serial and parallel binding of properties of fractional rheological models,. Each of the structures requires analysis, and it should be borne in mind that each regular (serial) connection of Newton's viscous element, fractional type or linear type, introduces one internal degree of freedom of movement of the ideal material model.

We will further analyze the behavior of some of the listed models and the structures of complex rheological models of ideal materials under conditions of rapid loading or holding at constant loads.

Let us analyze the component normal stresses and component axial dilations, as well as the rates of their changes, or through the structure of, in some of the complex rheological models of ideal materials, fractional type.

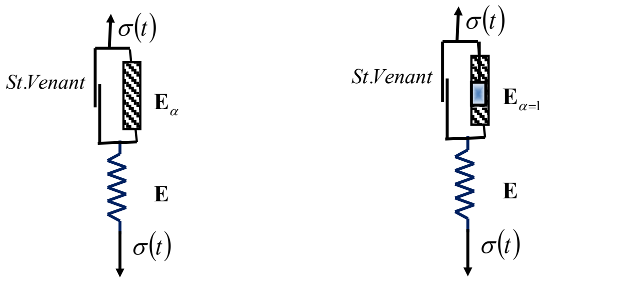
4.1 Generalized modified Bingham's model of ideal complex material, fractional type

* Generalized modified Bingham's model of complex ideal material, fractional type, with rheological structural formula $\boldsymbol{B}_{\alpha} = St.Venant/\boldsymbol{N}_{\alpha} - \boldsymbol{H}$, is shown in Figure 7. left.

The classic Bingham model of an ideal complex material (Figure 7. right) contains a parallel connection of the Saint Vemamt model of an ideally plastic material in parallel connection with Newton's model of an ideally viscous fluid, and this parallel connection is in turn connected with Hooke's model of an ideally elastic material. This connection determines the constitutive relations, that is, the equation of the connection between the normal stress and the axial dilation, that is, the rate of dilation in the form, see Fig. 9:

$$\sigma_z = \boldsymbol{E}\varepsilon_{z,1} \quad for \quad |\sigma_z| < |\sigma_{z,StV}| \qquad \varepsilon_{z,1}=\frac{\sigma_z}{\mathbf{E}}, for \quad |\sigma_z| < |\sigma_{z,StV}| \tag{31}$$

$$\sigma_z - \sigma_{z,StV} = \boldsymbol{E}_{\alpha=1}\dot{\varepsilon}_{z,}, \quad for \quad |\sigma_z| \geq |\sigma_{z,StV}| \tag{32}$$



$\mathbf{B}_\alpha = St.Venant/\mathbf{N}_\alpha - \mathbf{H}$ $\quad$ $\mathbf{B}_{\alpha=1} = St.Venant/\mathbf{N}_{\alpha=1} - \mathbf{H}$

Figure 9. Generalized Bingham model of an ideal complex material, fractional type (left) and the classic Bingham model of an ideal complex material (right)

The modified Bingham model of an ideal material, fractional type, contains a parallel connection of the Saint Venant model of an ideal plastic material, in parallel connection with a modified Newton model of an ideally viscous fluid, fractional type, and this parallel connection is in series connection with the Hooke's with this model of ideally elastic material. This connection determines the constitutive relations, i.e., the equations of the connection between normal stress and axial dilation, i.e., the rate of dilation, fractional type, in the form:

$$\sigma_z = \boldsymbol{E}\varepsilon_{z,1}, \quad for \quad |\sigma_z| < |\sigma_{z,StV}|, \varepsilon_{z,1} = \frac{\sigma_z}{\mathbf{E}}, for \quad |\sigma_z| < |\sigma_{z,StV}| \tag{33}$$

$$\sigma_z - \sigma_{z,StV} = \mathbf{E}_\alpha \mathsf{D}_t^\alpha[\varepsilon_z], \quad for \quad |\sigma_z| \geq |\sigma_{z,StV}| \tag{34}$$

The rate of dilation, fractional type, through this modified Bingham model of an ideal complex material, fractional type, is equal to:

$$\mathsf{D}_t^\alpha[\varepsilon_z] = \mathsf{D}_t^\alpha[\varepsilon_{z,1}] + \mathsf{D}_t^\alpha[\varepsilon_{z,2}] \tag{35}$$

and how it is:

$$\varepsilon_{z,1} = \frac{\sigma_z}{\mathbf{E}} \text{ thus, it follows } \mathsf{D}_t^\alpha[\varepsilon_{z.1}] = \frac{1}{\mathbf{E}}\mathsf{D}_t^\alpha[\sigma_z], for \quad |\sigma_z| < |\sigma_{z,StV}| \tag{36}$$

$$\mathsf{D}_t^\alpha[\varepsilon_z] = \frac{\sigma_z - \sigma_{z,StV}}{\mathbf{E}_\alpha}, \quad for \quad |\sigma_z| \geq |\sigma_{z,StV}| \tag{37}$$

And

$$\mathsf{D}_t^\alpha[\varepsilon_z] = \mathsf{D}_t^\alpha[\varepsilon_{z,1}] + \left\langle \mathsf{D}_t^\alpha[\varepsilon_{z,2}] \right\rangle_{=0} = \frac{1}{\mathbf{E}}\mathsf{D}_t^\alpha[\sigma_z], \quad for \quad |\sigma_z| < |\sigma_{z,StV}| \tag{38}$$

$$\mathsf{D}_t^\alpha[\varepsilon_z] = \mathsf{D}_t^\alpha[\varepsilon_{z,1}] + \mathsf{D}_t^\alpha[\varepsilon_{z,2}] = \frac{1}{\mathbf{E}}\mathsf{D}_t^\alpha[\sigma_z] + \frac{\sigma_z - \sigma_{z,StV}}{\mathbf{E}_\alpha}, \quad for \quad |\sigma_z| \geq |\sigma_{z,StV}| \tag{39}$$

This connection is the result of the fact that the model of Saint Venant's ideally plastic material is, by definition, rigid until the beginning of plastic yielding, i.e. up to a certain normal stress $\sigma_{z,StV}$, i.e. while the normal stress in the complex model is less than that value

$|\sigma_z| < |\sigma_{z,StV}|$, which is the condition of plasticity $|\sigma_z| < |\sigma_{z,StV}|$. When plastic flow begins, nominal stress remains constant and equals the value of plastic flow stress.

Until the beginning of plastic flow in the structural element, the model of Saint Venant's ideal plastic material, until the normal stress was reached in the modified Bingham's model of the ideal complex material, fractional type, the normal stress did not reach the value $\sigma_{z,StV}$, that is, while the normal stress in the complex model $|\sigma_z| < |\sigma_{z,StV}|$, model behaves like Hooke's model of an ideally elastic material. In that interval, the constitutive relationship $\sigma_z = \boldsymbol{E}\varepsilon_{z,1}, for |\sigma_z| < |\sigma_{z,StV}|$ between the normal stress and axial expansions is valid.

When plastic yielding has started in a structural element, in an element of Saint Venant's ideal plastic material, when further yielding continues in it, and when the normal stress is greater than the yield stress, that is $|\sigma_z| \geq |\sigma_{z,StV}|$, when , in the complete modified Bingham ideal complex material model, of fractional type, dilation rate, of fractional type, behaves according to relatiom:

$$\mathsf{D}_t^\alpha[\varepsilon_z] = \mathsf{D}_t^\alpha[\varepsilon_{z,1}] + \mathsf{D}_t^\alpha[\varepsilon_{z,2}] = \frac{1}{\mathbf{E}}\mathsf{D}_t^\alpha[\sigma_z] + \frac{\sigma_z - \sigma_{z,StV}}{\mathbf{E}_\alpha}, \quad for \quad |\sigma_z| \geq |\sigma_{z,StV}|. \tag{40}$$

We can conclude that the behavior in the complete modified Bingam model of an ideal complex material, fractional type, is in two phases. The first phase is the behavior of the model before reaching the normal yield stress $|\sigma_z| < |\sigma_{z,StV}|$, when it behaves like Hooke's model of an ideally elastic material, and then the constitutive relation applies $\sigma_z = \boldsymbol{E}\varepsilon_{z,1}, \quad for \quad |\sigma_z| < |\sigma_{z,StV}|$. In the second phase, the model behaves according to the constitutive relationship, the relationship between the rate of dilation of the fractional type and the normal stress in the form (40) and similar to the behavior of the modified Maxwell model, fractional type, for which we derived the constitutive relationship of the form (24) of the relationship between the rate of dilation, the fractional type and the normal stress. But those relations are different, but not identical, because the plastic deformations remain permanent, so when the normal stress decreases, new constitutive relations should be written.

But for this phase, according to the mathematical formalism, we can determine the characteristic behavior of this material- modified Bingham model of an ideal complex material, fractional type, in this phase of behavior, and that if we keep the material at a constant value of the axial dilation velocity, which tends to zero, until settling to a constant value, normal stress relaxation can occur in the material.

If we now assume that the behavior of the modified Bingam model of an ideal complex material, fractional type, is in the second phase (40), and that further the rate of dilation, fractional type, decreases and tends to zero $\mathsf{D}_t^\alpha[\varepsilon_z] \to 0$ and that we keep the element at that small rate of dilation, fractional type $\mathsf{D}_t^\alpha[\varepsilon_z]_0$, then the ordinary differential equation, of fractional order by normal stress, has the following form:

$$\mathsf{D}_t^\alpha[\sigma_z] + \frac{\mathbf{E}}{\mathbf{E}_\alpha}(\sigma_z - \sigma_{z,StV}) = \mathbf{E}\mathsf{D}_t^\alpha[\varepsilon_z]_0, \quad for \quad |\sigma_z| \geq |\sigma_{z,StV}| \tag{41}$$

$$\mathsf{D}_t^\alpha[\sigma_z] + \frac{\mathbf{E}}{\mathbf{E}_\alpha}\sigma_z = \mathbf{E}\mathsf{D}_t^\alpha[\varepsilon_z]_0 + \frac{\mathbf{E}}{\mathbf{E}_\alpha}\sigma_{z,StV}, \quad for \quad |\sigma_z| \geq |\sigma_{z,StV}| \tag{42}$$

Then, as in the previous consideration, applied the Laplace transform $\mathcal{L}$, so we get:

$$\mathsf{L}\{\mathsf{D}_t^{\alpha}[\sigma_z]\}+\frac{\mathbf{E}}{\mathbf{E}_\alpha}\mathsf{L}\{\sigma_z\}=\mathsf{L}\left\{\mathbf{E}\mathsf{D}_t^{\alpha}[\varepsilon_z]_0+\frac{\mathbf{E}}{\mathbf{E}_\alpha}\sigma_{z,StV}\right\}=\frac{1}{p}\left(\mathbf{E}\mathsf{D}_t^{\alpha}[\varepsilon_z]_0+\frac{\mathbf{E}}{\mathbf{E}_\alpha}\sigma_{z,StV}\right). \quad for \quad |\sigma_z|\geq|\sigma_{z,StV}| \tag{43}$$

that is,

$$\mathsf{L}\{\sigma_z\}=\frac{\left(\mathbf{E}_\alpha\mathsf{D}_t^{\alpha}[\varepsilon_z]_0+\sigma_{z,StV}\right)}{p\left(\frac{\mathbf{E}_\alpha}{\mathbf{E}}p^{\alpha}+1\right)},. \quad for \quad |\sigma_z|\geq|\sigma_{z,StV}| \tag{44}$$

After developing in the order of series of the complex number p , we get:

$$\mathsf{L}\{\sigma_z\}=\frac{\left(\mathbf{E}_\alpha\mathsf{D}_t^{\alpha}[\varepsilon_z]_0+\sigma_{z,SainrV}\right)}{p\left(\frac{\mathbf{E}_\alpha}{\mathbf{E}}p^{\alpha}+1\right)}\approx\left(\mathbf{E}_\alpha\mathsf{D}_t^{\alpha}[\varepsilon_z]_0+\sigma_{z,SainrV}\right)\left\langle\frac{1}{p}+\sum_{k=1}^{\infty}(-1)^k\left(\frac{\mathbf{E}_\alpha}{\mathbf{E}}\right)^k p^{k\alpha-1}\right\rangle,. \quad for \quad |\sigma_z|\geq|\sigma_{z,STrV}| \tag{45}$$

Using the inverse Laplace transform $\mathcal{L}^{-1}$, we get:

$$\sigma_z(t)=\mathsf{L}^{-1}\mathsf{L}\{\sigma_z\}\approx\left(\mathbf{E}_\alpha\mathsf{D}_t^{\alpha}[\varepsilon_z]_0+\sigma_{z,SainrV}\right)\cdot\mathsf{L}^{-1}\left\{\left\langle\frac{1}{p}+\sum_{k=1}^{\infty}(-1)^k\left(\frac{\mathbf{E}_\alpha}{\mathbf{E}}\right)^k p^{k\alpha-1}\right\rangle\right\}, \quad for \quad |\sigma_z|\geq|\sigma_{z,STV}| \tag{46}$$

Finally, the change of the normal stress at the points of the cross-sections of the model, in the second stress phase of the modified Bingam model of the ideal material, fractional type, when $|\sigma_z| \geq |\sigma_{z,StV}|$, that is, when plastic flow is present in the structural element, in the element of Saint Venant's ideal plastic material, we get normal stress as a function of time in the form:

$$\sigma_z(t)=\mathsf{L}^{-1}\mathsf{L}\{\sigma_z\}\approx\left(\mathbf{E}_\alpha\mathsf{D}_t^{\alpha}[\varepsilon_z]_0+\sigma_{z,SainrV}\right)\cdot\left\{1+\sum_{k=1}^{\infty}(-1)^k\left(\frac{\mathbf{E}_\alpha}{\mathbf{E}}\right)^k\frac{t^{(2-\alpha)k+1}}{\Gamma(2k+2-\alpha k)}\right\}, \quad for \quad |\sigma_z|\geq|\sigma_{z,STrV}| \tag{47}$$

This form of the previous expression (47) indicates that in this stress phase of the modified Bingam model of an ideal complex material, fractional type, the phenomenon of stress relaxation at constant dilation is present.

4.2 Generalized modified fractional Prandtl's model of ideal complex material.

Now, for example, let's consider, for example, a Prandtl body, which represents a body with the following properties: The material behaves like Hooke's ideally elastic material, as long as the normal stress is lower than the normal stress of plastic flow, and the total expansion consists of axial elastic expansion $\varepsilon_{z,H}$ and $\varepsilon_{z,K}$, $\varepsilon_{z,Sa\,int\,Vemamt}$ axial plastic dilatations:

$$\mathbf{H}* \qquad \varepsilon_{z,H}=\frac{1}{E}\sigma_{z,H} \tag{48}$$

and

$$\mathbf{P}* \quad \dot{\varepsilon}_{z,StV}=\frac{1}{P}\sigma_{z,StV} \tag{49}$$

Let the rate of dilation $\dot{\varepsilon}_{z,P}$ of plastic flow be determined by the following relation:

$$\dot{\varepsilon}_{z,P}=\frac{1}{E}\sigma_{z,H}+\frac{1}{2P}\sigma_{z,StV} \tag{50}$$

The normal yield stress and the axial dilatation at which yielding begins determine the yield point. For example, the flow is determined by introducing a certain hypothesis that is in better or weaker agreement with the experiment. There are different hypotheses about plastic flow.

4.3 Generalized modified fractional Lethersich's model of ideal complex material.

Generalized modified Lethersich model of ideal complex material, fractional type, shown in Figure 10 (left), is with rheological structural fatmule $\boldsymbol{L}_\alpha = \boldsymbol{H}/\boldsymbol{N}_\alpha - \boldsymbol{N}_\alpha$.

The classic Lethersich model of an ideal complex material, shown in Figure 8 (right), contains parallel connected models of Hooke's element of an ideally elastic material and Newton's element of an ideally viscous fluid (which form in parallel the Kelvin-Voigt model of an ideal material with the ability of subsequent elasticity) and in regular-serially connection with the model of Newton's element of an ideally viscous fluid.

Let us now denote by $\varepsilon_{z,N}$ and $\varepsilon_{z,K}$ the specific axial deformations - axial dilatations of Newton's model of an ideal viscous fluid and Kelvin-Voigt's model of an ideal solid elasto-viscous material, and for the total axial dilatation ε_z of Lethersich's model of an ideal complex material, which is an ordinal-serially connection of the Kelvin-Voigt model of an ideal body and Newton's model of an ideal viscous fluid, we get as in the sum of these axial dilatations: $\varepsilon_z = \varepsilon_{z,N} + \varepsilon_{z,K}$.

The constitutive relation -connection between the normal stress at the points of the cross-section of the complex model, axially stressed and axial dilations for Lethersich's model of an ideal material, is obtained from the relation of the sum of component velocities of axial dilations in the form: $\dot{\varepsilon}_z = \dot{\varepsilon}_{z,N} + \dot{\varepsilon}_{z,K}$. For Newton's model of an ideal viscous fluid, in the form:

$$\dot{\varepsilon}_{z,N} = \frac{\sigma_z}{\mathbf{E}_{\alpha=1}} \tag{51}$$

while for Kelvin-Voigt's ideal body model, the sum of normal stresses is in the form:

$$\sigma_{z,K} = \mathbf{E}\varepsilon_{z,K} + \mathbf{E}_{\alpha=1,1}\dot{\varepsilon}_{z,K} \qquad \dot{\varepsilon}_{z,N} = \frac{\sigma_z}{\mathbf{E}_{\alpha=1,2}} \tag{52}$$

It follows that:

$$\varepsilon_{z,K} = e^{-\frac{\mathbf{E}}{\mathbf{E}_{\alpha=1,1}}t}\left(\varepsilon_{z,K,o} + \frac{1}{\mathbf{E}_\alpha}\int_0^t e^{\frac{\mathbf{E}}{\mathbf{E}_{\alpha=1,1}}t}\sigma_{z,K}(t)dt\right) \tag{53}$$

in which $\varepsilon_{z,K,o}$ is the initial the axial dilation of Kelvin-Voigt's ideal model of an ideal body. Then differentiating the previous expression for the axial dilatation ion (53) of the Kelvin model of an ideal body and adding the axial dilatation rate $\dot{\varepsilon}_{z,N} = \frac{\sigma_z}{\boldsymbol{E}_{\alpha=1,2}}$ of the Newton's model of an ideal viscous fluid to the axial dilatation rate $\dot{\varepsilon}_{z,L}$ of the Lethersich model of an ideal complex material, we get the following expression:

$$\dot{\varepsilon}_{z.L} = \left(\frac{1}{\mathbf{E}_{\alpha=1,2}} + \frac{1}{\mathbf{E}_{\alpha=1,1}}\right)\sigma_z(t) - \frac{\mathbf{E}}{\mathbf{E}_{\alpha=1,,2}}e^{-\frac{\mathbf{E}}{\mathbf{E}_{\alpha=1,1}}t}\left(\varepsilon_{z,K,o} + \frac{1}{\mathbf{E}_\alpha}\int_0^t e^{\frac{\mathbf{E}}{\mathbf{E}_{\alpha=1,1}}t}\sigma_{z,K}(t)dt\right) \tag{54}$$

The modified Lethersich model of an ideal complex material, fractional type, which is shown in Figure, contains parallel connected models of Hooke's model of an ideally elastic material and Newton's model of an ideally viscous fluid, fractional type (which together form, in a paralel connection, a modified Kelvin's model of complex material, fractional type, with the ability of subsequent elasticity) and in serially connection with the Newtonian element model of an ideal viscous fluid, fractional type.

Let us now denote by $\varepsilon_{z,N}$ and $\varepsilon_{z,K}$ the specific axial deformations - axial dilations of the Newton's element of an ideally viscous fluid, fractional type, and the modified Kelvin model of an ideal complex solid elasto-viscous material, fractional type, and for the total axial dilatation $\varepsilon_z$of the modified Lethersich model of an ideal material, fractional type, (which is an serial connection of the modified Kelvin model of an ideal body of fractional type and Newton's model of an ideal viscous fluid, fractional type), we get, as in the sum of these axial dilatations: $\varepsilon_z = \varepsilon_{z,N} + \varepsilon_{z,K}$.

The constitutive relation-connection between normal stress and axial dilation for the modified Lethersich model of an ideal complex material, fractional type, is obtained from the relation of the sum of component velocities of axial dilations in the form: $\dot{\varepsilon}_{z,L} = \dot{\varepsilon}_{z,N} + \dot{\varepsilon}_{z,K}$, that is, $\mathsf{D}_t^\alpha\left[\varepsilon_{z,L}\right] = \mathsf{D}_t^\alpha\left[\varepsilon_{z,K}\right] + \mathsf{D}_t^\alpha\left[\varepsilon_{z,N}\right]$. For Newton's model of an ideal viscous fluid, fractional type is:

$$\mathsf{D}_t^\alpha\left[\varepsilon_{z,N}\right] = \frac{\sigma_z}{\mathbf{E}_\alpha} \tag{55}$$

while for the modified model of Kelvin's ideal model of the ideal body, fractional type, the sum of the voltages is:

$$\sigma_{z,K} = \mathbf{E}\varepsilon_{z,K} + \mathbf{E}_\alpha \mathsf{D}_t^\alpha\left[\varepsilon_{z,K}\right] \qquad \mathsf{D}_t^\alpha\left[\varepsilon_{z,K}\right] = \frac{\sigma_z}{\mathbf{E}_\alpha} \tag{56}$$

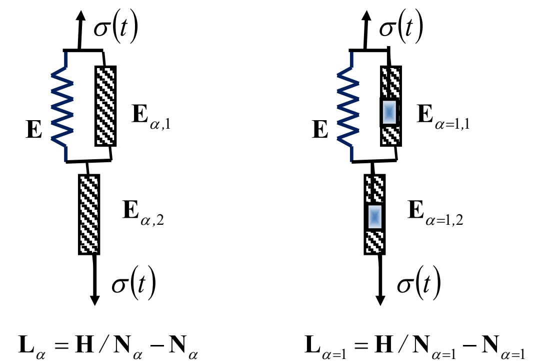


Figure 10. Generalized Lethersich's model of an ideal complex material, fractional type (left) and the classic Lethersich's model of an ideal complex material (right)

From this it follows that the axial dilation is the solution of the inhomogeneous equation of the fractional type:

$$\mathsf{D}_t^\alpha\left[\varepsilon_{z,K}\right] + \frac{\mathbf{E}}{\mathbf{E}_\alpha}\varepsilon_{z,K} = \frac{1}{\mathbf{E}_\alpha}\sigma_{z,K}(t) \tag{57}$$

Now, as in the previous consideration, apply the Laplace transform $\mathcal{L}$ and we get:

$$\mathsf{L}\left\{\mathsf{D}_t^{\alpha}\left[\varepsilon_{z,K}\right]\right\}+\frac{\mathbf{E}}{\mathbf{E}_\alpha}\mathsf{L}\left\{\varepsilon_{z,K}\right\}=\frac{1}{\mathbf{E}_\alpha}\mathsf{L}\left\{\sigma_{z,K}(t)\right\} \tag{58}$$

That is

$$\mathsf{L}\left\{\varepsilon_{z,K}\right\}=\frac{1}{\mathbf{E}_\alpha}\frac{1}{\left(\frac{\mathbf{E}}{\mathbf{E}_\alpha}+p^{\alpha}\right)}\mathsf{L}\left\{\sigma_{z,K}(t)\right\} \tag{59}$$

That is

$$\mathsf{L}\left\{\varepsilon_{z,K}\right\}=\frac{\mathbf{E}}{\left(1+\frac{\mathbf{E}_\alpha}{\mathbf{E}}p^{\alpha}\right)}\mathsf{L}\left\{\sigma_{z,K}(t)\right\} \tag{60}$$

On the right side, we have the product of two functions, which can be viewed as the product of two Laplace transforms, two functions that are in convolution. Therefore, we need to determine the inverse Laplace transformations of each of these functions individually: $\frac{E}{\left(1+\frac{E_\alpha}{E}p^{\alpha}\right)}$ and $\mathsf{L}\left\{\sigma_{z,K}(t)\right\}$.

$$\mathsf{L}^{-1}\left\{\frac{\mathbf{E}}{\left(1+\frac{\mathbf{E}_\alpha}{\mathbf{E}}p^{\alpha}\right)}\right\}=\mathbf{E}\mathsf{L}^{-1}K=1\left\{1+\sum_{K=1}^{\infty}(-1)^k\left(\frac{\mathbf{E}_\alpha}{\mathbf{E}}\right)^k p^{k\alpha}\right\}=\mathbf{E}\sum_{k=1}^{\infty}(-1)^k\left(\frac{\mathbf{E}_\alpha}{\mathbf{E}}\right)^k\frac{t^{(2-\alpha)k+2}}{\Gamma(2k+1-\alpha k)} \tag{61}$$

$$\mathsf{L}^{-1}\mathsf{L}\left\{\sigma_{z,K}(t)\right\}=\sigma_{z,K}(t) \tag{62}$$

The solution is the convolution integral:

$$\varepsilon_{z,K}=\mathbf{E}\int_0^t\sigma_{z,K}(t-\tau)\sum_{k=1}^{\infty}(-1)^k\left(\frac{\mathbf{E}_\alpha}{\mathbf{E}}\right)^k\frac{t^{(2-\alpha)k}}{\Gamma(2k+1-\alpha k)}d\tau \tag{63}$$

which $\varepsilon_{z,K,o}$ is the initial axial dilatation of the modified Kelvin-Voigt's ideal model of an complex ideal body, fractional type. Then differentiating the previous expression for the axial dilatation (63) of Kelvin-Voigt's model of an ideal complex material body and adding the velocity of axial dilatation $\dot{\varepsilon}_{z,N}=\frac{\sigma_z}{E_\alpha}$ of Newton's model of an ideal viscous fluid, fractional type, to the velocity of axial dilatation of Lethersich's model of an ideal complex material, we get the following expression: $\mathsf{D}_t^{\alpha}\left[\varepsilon_{z,L}\right]=\mathsf{D}_t^{\alpha}\left[\varepsilon_{z,K}\right]+\mathsf{D}_t^{\alpha}\left[\varepsilon_{z,N}\right]$.

As can be seen from the structure of thegeneralized modified Lethersich model of an ideal complex material, fractional type, that it is formed by the serial connection of the modified fractional Kelvin-Voigt's model of an ideal body, and fractional Newton's model of an ideal viscous fluid, and as we have shown that the models of these ideal materials, of the fractional type, have properties, the first of which is elastic, and the second is the property of stress relaxation. This means that the modified fractional Lethersich model of an ideal material possesses both these properties and subsequent elasticity and normal stress relaxation, and that it is a more complex material model than its substructure.

4.4 Generalized modified fractional Jeffrey's model of ideal complex material.

Generalized modified Jeffrey's model of ideal material, fractional type, is with rheological structural formula in the form $\boldsymbol{J}_\alpha = \boldsymbol{N}_\alpha/(\boldsymbol{H}-\boldsymbol{N}_\alpha)$, and with structure which is shown in the Fig. 11 (left).

Classical Jeffreys' model of an ideal complex material, shown in the Fig. 9 (right), contains a parallel connection of Newton's element of an ideally viscous fluid and Maxwell's model of an ideally complex visco-elastic material, for which there are individual constitutive relations of normal stress and axial dilation, i.e. dilation rate, in the form:

$$\dot{\varepsilon}_{z,N1} = \frac{\sigma_{z,N}}{\mathbf{E}_{\alpha=1,1}} \qquad \sigma_{z,N1} = \mathbf{E}_{\alpha=1,1}\dot{\varepsilon}_{z,N1} \tag{64}$$

$$\dot{\varepsilon}_{z,M} = \frac{\dot{\sigma}_{z,M}}{\mathbf{E}} + \frac{\sigma_{z,M}}{\mathbf{E}_{\alpha=1,2}} \tag{65}$$

From here, by integration, it follows that (see Reference [1]):

$$\sigma_{z,M} = e^{-\frac{\mathbf{E}}{\mathbf{E}_{\alpha=1,2}}t}\left(\sigma_{z,M,o} + \mathbf{E}_{\alpha=1,2}\int_0^t e^{\frac{\mathbf{E}}{\mathbf{E}_{\alpha=1,2}}t}\dot{\varepsilon}_{z,M}(t)dt\right) \tag{66}$$

Then, by adding $\sigma_{z,N1}$ and $\sigma_{z,M}$, we get the resulting normal stress $\sigma_{z,J}$ in classical Jeffrey's ideal complec material model and form:

$$\sigma_{z,J} = \sigma_{z,N1} + \sigma_{z,M} = \mathbf{E}_{\alpha=1,1}\dot{\varepsilon}_{z,N1} + e^{-\frac{\mathbf{E}}{\mathbf{E}_{\alpha=1,2}}t}\left(\sigma_{z,M,o} + \mathbf{E}_{\alpha=1,2}\int_0^t e^{\frac{\mathbf{E}}{\mathbf{E}_{\alpha=1,2}}t}\dot{\varepsilon}_{z,M}(t)dt\right) \tag{67}$$

and it is expressed through the rate of axial dilatation.

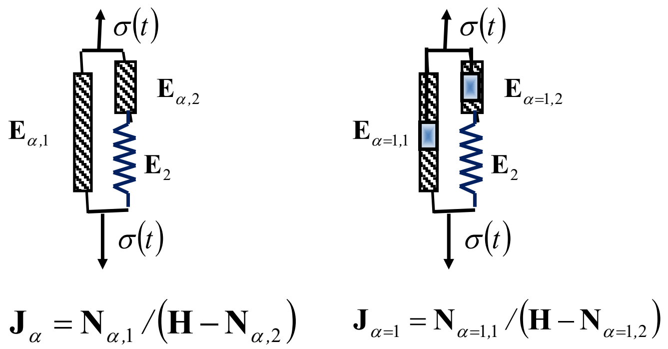


Figure 11. Generalized Jeffrey's model of an ideal complex material, fractional type (left) and the classic Jeffrys' model of an ideal complex material (right)

The modified Jeffrey's model of an ideal complex material, fractional type, shown in Fig. 11, contains a parallel connection of Newton's element of an ideally viscous fluid, fractional type, and a modified fractional Maxwell model of an ideally visco-elastic complex material, for which there are individually constitutive relations of normal stress and axial dilation, i.e. rate of axial dilation, fractional type, in the form:

$$\mathsf{D}_t^{\alpha}\left[\varepsilon_{z,N1}\right]=\frac{\sigma_{z,N}}{\mathbf{E}_{\alpha,1}} \qquad \sigma_{z,N1}=\mathbf{E}_{\alpha 1}\mathsf{D}_t^{\alpha}\left[\varepsilon_{z,N1}\right] \tag{68}$$

$$\mathsf{D}_t^{\alpha}\left[\varepsilon_{z,M}\right]=\frac{\mathsf{D}_t^{\alpha}\left[\sigma_{z,M}\right]}{\mathbf{E}}+\frac{\sigma_{z,M}}{\mathbf{E}_{\alpha,2}} \tag{69}$$

From the previous relation (69), we form an ordinary inhomogeneous fractional order differential equation of the form:

$$\mathsf{D}_t^{\alpha}\left[\sigma_{z,M}\right]+\frac{\mathbf{E}\sigma_{z,M}}{\mathbf{E}_{\alpha,2}}=\mathbf{E}\mathsf{D}_t^{\alpha}\left[\varepsilon_{z,M}(t)\right] \tag{70}$$

The solution of the previous (70) inhomogeneous differential equation of fractional order is solved using the Laplace transform $\mathcal{L}$, then by developing it in series, and returning it to the time domain with the inverse Laplace transform $\mathcal{L}^{-1}$, from which it follows:

$$\mathsf{L}\left\{\mathsf{D}_t^{\alpha}\left[\sigma_{z,M}\right]\right\}+\frac{\mathbf{E}\mathsf{L}\left\{\sigma_{z,M}\right\}}{\mathbf{E}_{\alpha,2}}=\mathbf{E}\mathsf{L}\left\{\mathsf{D}_t^{\alpha}\left[\varepsilon_{z,M}(t)\right]\right\} \tag{71}$$

And

$$\mathsf{L}\left\{\sigma_{z,M}\right\}=\mathbf{E}_{\alpha,2}\mathsf{L}\left\{\mathsf{D}_t^{\alpha}\left[\varepsilon_{z,M}(t)\right]\right\}\frac{1}{\frac{\mathbf{E}_{\alpha,2}}{\mathbf{E}}p^{\alpha}+1} \tag{72}$$

Analytical approximation of previous expression (72) is in the following form:

$$\mathsf{L}\left\{\sigma_{z,M}\right\}=\mathbf{E}_{\alpha,2}\mathsf{L}\left\{\mathsf{D}_t^{\alpha}\left[\varepsilon_{z,M}(t)\right]\right\}\frac{1}{\frac{\mathbf{E}_{\alpha,2}}{\mathbf{E}}p^{\alpha}+1}\approx\mathbf{E}_{\alpha,2}\mathsf{L}\left\{\mathsf{D}_t^{\alpha}\left[\varepsilon_{z,M}(t)\right]\right\}\left\langle 1+\sum_{k=1}^{\infty}(-1)^k\left(\frac{\mathbf{E}_{\alpha,2}}{\mathbf{E}}\right)^k p^{\alpha k}\right\rangle \tag{73}$$

$$\mathsf{L}\left\{\sigma_{z,M}\right\}=\approx\mathbf{E}_{\alpha=1,2}\mathsf{L}\left\{\mathsf{D}_t^{\alpha}\left[\varepsilon_{z,M}(t)\right]\right\}\left\langle 1+\sum_{k=1}^{\infty}(-1)^k\left(\frac{\mathbf{E}_{\alpha,2}}{\mathbf{E}}\right)^k p^{\alpha k}\right\rangle$$

Inverse Laplace transform expressed in the form:

$$\mathsf{L}^{-1}\mathsf{L}\left\{\sigma_{z,M}\right\}=\approx\mathbf{E}_{\alpha=1,2}\mathsf{L}^{-1}\mathsf{L}\left\{\mathsf{D}_t^{\alpha}\left[\varepsilon_{z,M}(t)\right]\right\}\mathsf{L}^{-1}\left\{\left\langle 1+\sum_{k=1}^{\infty}(-1)^k\left(\frac{\mathbf{E}_{\alpha=1,2}}{\mathbf{E}}\right)^k p^{\alpha k}\right\rangle\right\} \tag{74}$$

gives an approximate analytical expression of the normal stress in function of axial dilatation of the Maxwell substructure model in the form (30) and by using integral of convolution in the form:

$$\sigma_{z,M}=\approx\mathbf{E}_{\alpha=1,2}\int_0^t\mathsf{D}_t^{\alpha}\left[\varepsilon_{z,M}(t-\tau)\right]\sum_{k=0}^{\infty}(-1)^k\left(\frac{\mathbf{E}_{\alpha,2}}{\mathbf{E}}\right)^k\frac{t^{(2-\alpha)kk}}{\Gamma(2k+1-\alpha k)}d\tau \tag{75}$$

Because inverse Laplace transform gives expression in time domain:

$$\mathsf{L}^{-1}\left\{\left\langle 1+\sum_{k=1}^{\infty}(-1)^k\left(\frac{\mathbf{E}_{\alpha=1,2}}{\mathbf{E}}\right)^k p^{\alpha k}\right\rangle\right\}=\mathsf{L}^{-1}\left\{\left\langle\sum_{k=0}^{\infty}(-1)^k\left(\frac{\mathbf{E}_{\alpha=1,2}}{\mathbf{E}}\right)^k p^{\alpha k}\right\rangle\right\}=\sum_{k=0}^{\infty}(-1)^k\left(\frac{\mathbf{E}_{\alpha=1,2}}{\mathbf{E}}\right)^k\frac{t^{(2-\alpha)kk}}{\Gamma(2k+1-\alpha k)} \tag{76}$$

Then by adding $\sigma_{z,N1}$ and $\sigma_{z,M}$, we get the resulting normal stress $\sigma_{z,J}$ in the modified Jeffrys model $\boldsymbol{J}_{\alpha}$ of an ideal complex material, frakctional type, and in the followung form:

$$\sigma_{z,J} = \sigma_{z,N1} + \sigma_{z,M} = \mathbf{E}_{\alpha,1} \mathsf{D}_t^{\alpha}\left[\varepsilon_{z,N1}\right] + \mathbf{E}_{\alpha,2} \int_0^t \mathsf{D}_t^{\alpha}\left[\varepsilon_{z,M}(t-\tau)\right] \sum_{k=0}^{\infty} (-1)^k \left(\frac{\mathbf{E}_{\alpha,2}}{\mathbf{E}}\right)^k \frac{t^{(2-\alpha)kk}}{\Gamma(2k+1-\alpha k)} d \tag{77}$$

And finally, aproximate analytical expression of mormal stress in function of the the fractional type velocity of axial dilation of a generalized-modigied complex Jeffrys' model of an ideal complex material, fractional type. Obtained aproximate analitical expression – diferential, fractional order, relation (77) is constitutive relation, differential, fractional order, form.

4.5 Generalized rheological modified Burgers' model of ideal complex material, fractional type.

Generalized modified Burgers model $\boldsymbol{B}_{u,\alpha}$ of ideal complex material, fractional type, it show it in the Figure 12 (left size), with rheological structural formule $\boldsymbol{B}_{u,\alpha} = \boldsymbol{K} - \boldsymbol{M}$ or $\boldsymbol{B}_{u,\alpha} = (\boldsymbol{H}/\boldsymbol{N}_{\alpha}) - (\boldsymbol{H} - \boldsymbol{N}_{\alpha})$.

The classic Burgers' model $\boldsymbol{B}_{u,\alpha=1}$ of an ideal complex material, it show it in the Figure 10 (right size), and it represents an ordinal-serially connection of Maxwell's and Kelvin-Voigt's complex models of ideal materials. The constitutive relations (constituve equations) of this complex material model contain higher-order time derivatives. The total axial dilatation of the complex model of the classical Burgers model $\boldsymbol{B}_{u,\alpha=1}$ of an ideal complex material is equal to the sum of the component dilatations $\varepsilon_{z,M}$ and $\varepsilon_{z,K}$, also complex models of Maxwell's and Kelvin-Voigt's basic complex models of ideal complex materials, while the normal stress in all points of cross sections of the component complex models is equal. Based on this analysis and derived conclusions, we can write the following relations:

$$\varepsilon_{z,Bu} = \varepsilon_{z,M} + \varepsilon_{z,K} \tag{78}$$

$$\sigma_{z,K1} = \mathbf{E}_K \varepsilon_{z,K} + \mathbf{E}_{\alpha=1,2,,K} \dot{\varepsilon}_{z,K} \tag{79}$$

$$\dot{\varepsilon}_{z,M} = \frac{\dot{\sigma}_{z,M}}{\mathbf{E}_M} + \frac{\sigma_{z,M}}{\mathbf{E}_{\alpha=1,1,M}} \tag{80}$$

Let's differentiate the first constitutive relation from the previous system (78), and replace the terms from the third constitutive equation of the previous system, and based on that we could write the following constitutive relation for the classic Burgers model $\boldsymbol{B}_{u,\alpha=1}$ of an ideal complex material:

$$\dot{\varepsilon}_{z,Bu} = \dot{\varepsilon}_{z,M} + \dot{\varepsilon}_{z,K} = \frac{\dot{\sigma}_{z,M}}{\mathbf{E}_M} + \frac{\sigma_{z,M}}{\mathbf{E}_{\alpha=1,1,M}} + \dot{\varepsilon}_{z,K} \tag{81}$$

Nor, we put the last constitutive equation (81) differentiated once more in time, so we get:

$$\ddot{\varepsilon}_{z,Bu} = \frac{\ddot{\sigma}_{z,M}}{\mathbf{E}_M} + \frac{\dot{\sigma}_{z,M}}{\mathbf{E}_{\alpha=1,1,M}} + \ddot{\varepsilon}_{z,K} \tag{82}$$

Let us now multiply the relation (80) by $\boldsymbol{E}_K$, and the last relation (82) by $\boldsymbol{E}_{\alpha=1,M}$, and add them with $\dot{\sigma}_{z,K}=\mathbf{E}_K\dot{\varepsilon}_{z,K}+\mathbf{E}_{\alpha=1,2,K}\ddot{\varepsilon}_{z,K}$ and for the component Kelvin-Voigt complex model of an ideal complex material, so that in the result we derive the following relation:

$$\ddot{\sigma}_{z,Bu}+\dot{\sigma}_{z,Bu}\left(\frac{\mathbf{E}_K}{\mathbf{E}_{\alpha=1,1,M}}+\frac{\mathbf{E}_M}{\mathbf{E}_{\alpha=1,2,K}}+\frac{\mathbf{E}_M}{\mathbf{E}_{\alpha=1,M1,}}\right)+\sigma_{z,Bu}\frac{\mathbf{E}_M}{\mathbf{E}_{\alpha=1,1,M}}\frac{\mathbf{E}_K}{\mathbf{E}_{\alpha=1,2,K}}=\frac{\mathbf{E}_K\mathbf{E}_M}{\mathbf{E}_{\alpha=1,2,K}}\dot{\varepsilon}_{z,Bu}+\mathbf{E}_M\ddot{\varepsilon}_{z,Bu} \tag{83}$$

This last relationship between normal stress $\sigma_{z,Bu}$ and axial dilatation $\varepsilon_{z,Bu}$, i.e. the corresponding views, is the constitutive relationship of the ideal rheological material model of the structure of the classic Burgers model $\boldsymbol{B}_{u,\alpha=1}$ of ideal complex material, which is shown in Fig. 10 (right size). Let us recall that it represents the serial relation of Maxwell's and Kelvin-Voigt's complex models of ideal basic complex materials. And under certain conditions, it has properties of subsequent elasticity and or relaxation of normal stress.

Generalized rheological modified Burgers model $\boldsymbol{B}_{u,\alpha}$ of ideal complex material, fractional type, with rheological structural fatmule $\boldsymbol{B}_{u,\alpha}=\boldsymbol{K}-\boldsymbol{M}$ or$\boldsymbol{B}_{u,\alpha}=(\boldsymbol{H}/\boldsymbol{N}_\alpha)-(\boldsymbol{H}-\boldsymbol{N}_\alpha)$.

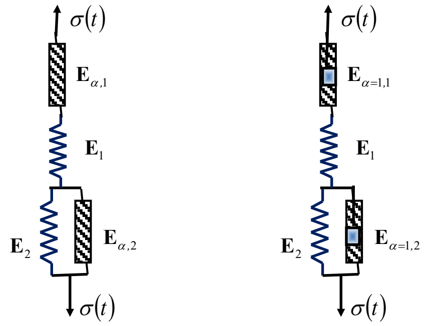
A modified Burgers model $\boldsymbol{B}_{u,\alpha}$of an ideal complex material, fractional type, is shown in Fig. 10 (left-hand size), and represents an serially connection of Maxwell's complex ideal material, fractional type, and Kelvin-Voigt's complex model of an ideal complex material, fractional type. The constitutive relations (equations of connections between normal stress and axial dilatation) of this complex material model, of the fractional type, contain derivatives of the fractional order in time, and also of the higher order. The total dilatation of the complex model of the rheological Burgers ideal complex material model, fractional type $\boldsymbol{B}_{u,\alpha}$, is equal to the sum of the component dilatations $\varepsilon_{z,M}$ and $\varepsilon_{z,K}$ and the component, also, complex models of the Maxwell and Kelvin-Voigt complex models of ideal materials, fractional type, while the normal stress in all points of all cross- sections of the component complex model equal to $\sigma_{z,K1}=\sigma_{z,M}=\sigma_{z,Bu}$. Based on this analysis and derived conclusions, we can write the following relations:

$$\varepsilon_{z,Bu}=\varepsilon_{z,M}+\varepsilon_{z,K}$$

$$\mathsf{D}_t^\alpha\left[\varepsilon_{z,Bu}\right]=\mathsf{D}_t^\alpha\left[\varepsilon_{z,M}\right]+\mathsf{D}_t^\alpha\left[\varepsilon_{z,K}\right] \tag{84}$$

$$\sigma_{z,K1}=\mathbf{E}_K\varepsilon_{z,K}+\mathbf{E}_{\alpha,2,K}\mathsf{D}_t^\alpha\left[\varepsilon_{z,K}\right] \tag{85}$$

$$\mathsf{D}_t^\alpha\left[\varepsilon_{z,M}\right]=\frac{\mathsf{D}_t^\alpha\left[\sigma_{z,M}\right]}{\mathbf{E}_M}+\frac{\sigma_{z,M}}{\mathbf{E}_{\alpha,1,M}} \tag{86}$$



$\mathbf{B}_{u,\alpha} = \mathbf{K} - M$ $\mathbf{B}_{u,\alpha=1} = \mathbf{K}_{\alpha=1} - M_{\alpha=1}$

$\mathbf{B}_{u,\alpha} = (\mathbf{H}/\mathbf{N}_{\alpha}) - (\mathbf{H} - \mathbf{N}_{\alpha})$ $\mathbf{B}_{u,\alpha=1} = (\mathbf{H}/\mathbf{N}_{\alpha=1}) - (\mathbf{H} - \mathbf{N}_{\alpha=1})$

Figure 12. Generalized Burgers' model of an ideal complex material, fractional type (left) and the classic Burgers' model of an ideal complex material (right)

Let's differentiate the first constitutive relation, using the differential operator of fractional order $\mathsf{D}_t^{\alpha}[\varepsilon_{z,K}]$ from the previous system (7), and replace the terms from the third constitutive equation of the previous system, and based on that we could write the following constitutive relation for the rheological modified Burgers model $\boldsymbol{B}_{u,\alpha}$ of an ideal complex material, fractional type, in the form:

$$\mathsf{D}_t^{\alpha}\left[\varepsilon_{z,Bu}\right] = \mathsf{D}_t^{\alpha}\left[\varepsilon_{z,M}\right] + \mathsf{D}_t^{\alpha}\left[\varepsilon_{z,K}\right] = \frac{\mathsf{D}_t^{\alpha}\left[\sigma_{z,M}\right]}{\mathbf{E}_M} + \frac{\sigma_{z,M}}{\mathbf{E}_{\alpha,,1,M}} + \mathsf{D}_t^{\alpha}\left[\varepsilon_{z,K}\right] \quad (87)$$

or

$$\frac{\mathsf{D}_t^{\alpha}\left[\sigma_{z,M}\right]}{\mathbf{E}_M} + \frac{\sigma_{z,M}}{\mathbf{E}_{\alpha,1,M}} = \mathsf{D}_t^{\alpha}\left[\varepsilon_{z,Bu}\right] - \mathsf{D}_t^{\alpha}\left[\varepsilon_{z,K}\right] \quad (88)$$

$$\sigma_{z,K1} = \mathbf{E}_K \varepsilon_{z,K} + \mathbf{E}_{\alpha,2,,K} \mathsf{D}_t^{\alpha}\left[\varepsilon_{z,K}\right] = \sigma_{z,M} = \sigma_{z,Bu} \quad (89)$$

That is, as throughout the modified structure of the rheological Burgers model $\boldsymbol{B}_{u,\alpha}$ of an ideal complex material, fractional type, the normal stress is equal to $\sigma_{z,K1} = \sigma_{z,M} = \sigma_{z,bu}$, so the previous relations - equations of the fractional order can be written in the following form:

$$\frac{\mathsf{D}_t^{\alpha}\left[\sigma_{z,Bu}\right]}{\mathbf{E}_M} + \frac{\sigma_{z,Bu}}{\mathbf{E}_{\alpha,1,M}} = \mathsf{D}_t^{\alpha}\left[\varepsilon_{z,Bu}\right] - \mathsf{D}_t^{\alpha}\left[\varepsilon_{z,K}\right] \quad (90)$$

$$\sigma_{z,Bu} = \mathbf{E}_K \varepsilon_{z,K} + \mathbf{E}_{\alpha,2,K} \mathsf{D}_t^{\alpha}\left[\varepsilon_{z,K}\right] \quad (91)$$

These last two relations between normal stress $\sigma_{z,Bu}$ and axial dilatation $\varepsilon_{z,Bu}$, i.e. the corresponding derivatives, fractional order, are coupled differential constitutive relations, fractional orders, model of ideal rheological material, modified structure of Burgers model $\boldsymbol{B}_{u,\alpha}$ of ideal modified Burgers model of ideal complex material, fractional order type, shown in Fig. 10 (left-hand size). That model of the modified rheological Burgers' model

$\boldsymbol{B}_{u,\alpha}$ of ideal complex material, fractional type, and represents the serially connection of Maxwell's and Kelvin-Voigt's complex models of ideal materials, fractional type. And under certain conditions, it has properties of subsequent elasticity and or relaxation of normal stress.

From this system of differential constitutive relations, of fractional order, we can eliminate the rate (velocity) of dilation of this type and obtain only one relation of the connection of normal stress and axial dilation of the modified structure, Burgers' model of an ideal material, fractional type $\boldsymbol{B}_{u,\alpha}$, but by moving into the domain of the Laplace transform in the following form:

$$\mathsf{D}_t^{\alpha}\left[\varepsilon_{z,Bu}\right]=\mathsf{D}_t^{\alpha}\left[\varepsilon_{z,M}\right]+\mathsf{D}_t^{\alpha}\left[\varepsilon_{z,K}\right]=\frac{\mathsf{D}_t^{\alpha}\left[\sigma_{z,M}\right]}{\mathbf{E}_M}+\frac{\sigma_{z,M}}{\mathbf{E}_{\alpha,,1,M}}+\mathsf{D}_t^{\alpha}\left[\varepsilon_{z,K}\right] \quad (92)$$

$$\frac{\mathsf{D}_t^{\alpha}\left[\sigma_{z,Bu}\right]}{\mathbf{E}_M}+\frac{\sigma_{z,Bu}}{\mathbf{E}_{\alpha,2,,M}}=\mathsf{D}_t^{\alpha}\left[\varepsilon_{z,Bu}\right]-\mathsf{D}_t^{\alpha}\left[\varepsilon_{z,K}\right] \quad (93)$$

$$\sigma_{z,Bu}=\mathbf{E}_K\varepsilon_{z,K}+\mathbf{E}_{\alpha,2,K}\mathsf{D}_t^{\alpha}\left[\varepsilon_{z,K}\right] \quad (94)$$

$$\mathsf{L}\left\{\mathsf{D}_t^{\alpha}\left[\sigma_{z,Bu}\right]\right\}+\frac{\mathbf{E}_M}{\mathbf{E}_{\alpha,1,M}}\mathsf{L}\left\{\sigma_{z,Bu}\right\}=\mathbf{E}_M\mathsf{L}\left\{\mathsf{D}_t^{\alpha}\left[\varepsilon_{z,Bu}\right]\right\}-\mathbf{E}_M\mathsf{L}\left\{\mathsf{D}_t^{\alpha}\left[\varepsilon_{z,K}\right]\right\} \quad (95)$$

$$\mathsf{L}\left\{\sigma_{z,Bu}\right\}=\mathbf{E}_K\mathsf{L}\left\{\varepsilon_{z,K}\right\}+\mathbf{E}_{\alpha,2,K}\mathsf{L}\left\{\mathsf{D}_t^{\alpha}\left[\varepsilon_{z,K}\right]\right\} \quad (96)$$

or

$$\mathsf{L}\left\{\sigma_{z,Bu}\right\}\left\langle p^{\alpha}+\frac{\mathbf{E}_M}{\mathbf{E}_{\alpha,1,M}}\right\rangle=\mathbf{E}_M p^{\alpha}\left\langle\mathsf{L}\left\{\varepsilon_{z,Bu}\right\}-\mathsf{L}\left\{\varepsilon_{z,K}\right\}\right\rangle \quad (97)$$

$$\mathsf{L}\left\{\sigma_{z,Bu}\right\}=\mathbf{E}_K\mathsf{L}\left\{\varepsilon_{z,K}\right\}+\mathbf{E}_{\alpha,2,K}p^{\alpha}\mathsf{L}\left\{\varepsilon_{z,K}\right\} \quad (98)$$

or

$$\mathsf{L}\left\{\sigma_{z,Bu}\right\}\left\langle p^{\alpha}+\frac{\mathbf{E}_M}{\mathbf{E}_{\alpha,1,M}}\right\rangle=\mathbf{E}_M p^{\alpha}\left\langle\mathsf{L}\left\{\varepsilon_{z,Bu}\right\}-\mathsf{L}\left\{\varepsilon_{z,K}\right\}\right\rangle \quad (99)$$

$$\mathsf{L}\left\{\varepsilon_{z,K}\right\}=\frac{1}{\left(\mathbf{E}_K+\mathbf{E}_{\alpha,2,K}p^{\alpha}\right)}\mathsf{L}\left\{\sigma_{z,Bu}\right\} \quad (100)$$

Now let's replace the last expression in (99) and get one equation:

$$\mathsf{L}\left\{\sigma_{z,Bu}\right\}\left\langle p^{\alpha}+\frac{\mathbf{E}_M}{\mathbf{E}_{\alpha,1,M}}\right\rangle=\mathbf{E}_M p^{\alpha}\left\langle\mathsf{L}\left\{\varepsilon_{z,Bu}\right\}-\frac{1}{\left(\mathbf{E}_K+\mathbf{E}_{\alpha,2,K}p^{\alpha}\right)}\mathsf{L}\left\{\sigma_{z,Bu}\right\}\right\rangle \quad (101)$$

Or in the form

$$\mathsf{L}\left\{\sigma_{z,Bu}\right\}\left\langle p^{\alpha}+\frac{\mathbf{E}_M}{\mathbf{E}_{\alpha,1,M}}+p^{\alpha}\frac{\mathbf{E}_M}{\left(\mathbf{E}_K+\mathbf{E}_{\alpha,2,K}p^{\alpha}\right)}\right\rangle=\mathbf{E}_M p^{\alpha}\mathsf{L}\left\{\varepsilon_{z,Bu}\right\} \quad (102)$$

$$\mathrm{L}\{\sigma_{z,Bu}\} = \mathbf{E}_M \mathrm{L}\{\varepsilon_{z,Bu}\} \frac{p^\alpha}{\left\langle p^\alpha \left[1 + \frac{\mathbf{E}_M}{\left(\mathbf{E}_K + \mathbf{E}_{\alpha,2,K} p^\alpha\right)} \right] + \frac{\mathbf{E}_M}{\mathbf{E}_{\alpha,1,M}} \right\rangle} \tag{103}$$

$$\mathrm{L}\{\sigma_{z,Bu}\} = \mathbf{E}_M \mathrm{L}\{\varepsilon_{z,Bu}\} \frac{\left(\mathbf{E}_K + \mathbf{E}_{\alpha,2,K} p^\alpha\right) p^\alpha}{\left\langle \mathbf{E}_M p^\alpha \left[\frac{\left(\mathbf{E}_K + \mathbf{E}_{\alpha,2,K} p^\alpha\right)}{\mathbf{E}_M} + 1 \right] + \frac{\left(\mathbf{E}_K + \mathbf{E}_{\alpha,2,K} p^\alpha\right)}{\mathbf{E}_{\alpha,1,M}} \right\rangle} \tag{104}$$

$$\mathrm{L}\{\sigma_{z,Bu}\} = \mathbf{E}_M \mathrm{L}\{\varepsilon_{z,Bu}\} \frac{\left(\mathbf{E}_K + \mathbf{E}_{\alpha,2,K} p^\alpha\right) p^\alpha}{\left\langle \mathbf{E}_M p^\alpha \left[\left(\mathbf{E}_K + \mathbf{E}_{\alpha,2,K} p^\alpha\right) \left(\frac{1}{\mathbf{E}_M} + \frac{1}{\mathbf{E}_{\alpha,1,M}} \right) + 1 \right] \right\rangle} \tag{105}$$

And now, by applying the inverse Laplace transform, it is possible to obtain the integral convoluting the time function of the normal stress $\sigma_{z,Bu}$ from the axial dilatation $\varepsilon_{z,Bu}$ of the structure of the Burgers model of an ideal material, fractional type $\boldsymbol{B}_{u,\alpha}$.

5. CONCLUDING REMARKS

In chapters I, II and III of the paper, the author introduces two new rheological elements. One of the elements is a generalized Newtonian model of an ideal viscous fluid of the fractional type, characterized by a constitutive relation involving a fractional-order differential operator, where the order of differentiation lies within the interval (0, 1). The other ideal element is a Faraday-type element with piezoelectric properties, described by fractional-order differential constitutive relations. These relate the normal stress and axial dilatation to the electric voltage of the polarization field, or equivalently, to the dielectric displacement. This element represents a coupling between mechanical and electrical fields, involving tensors that describe the mechanical and electrical states of the piezoelectric material.

It is shown that the generalized Newton element of an ideal viscous fluid, fractional type, has the ability to dissipate mechanical energy, fractional type.

Then the structures of new complex models of ideal materials, of the fractional type, are set up by replacing in the classic models of ideal materials, the classic Newton element of an ideal fluid, replacing it with a generalized Newton element of an ideal viscous fluid, of the fractional type. For each new generalized-modified model of a complex ideal material, of the fractional type, constitutive fractional differential relations are set. For the basic generalized-modified fractional Kelvin-Voigt and Maxwell models it is shown that the former exhibits the property of subsequent elasticity, while the latter demonstrates normal stress relaxation.

Then, the next task is to define standard light complex models, of the fractional type, and study rheological dynamic systems, also of the fractional type, in which these standard light models act as connection elements in the rheological dynamic system.

It is assumed that there are rheological dynamic systems, of the fractional type, oscillator type or crawler type, depending on the properties of standard light rheological models, of the fractional type.

In this paper, a new class of rheological models of ideal materials of the fractional type has been developed, based on the introduction of a generalized-modified Newtonian element of an ideal viscous fluid, characterized by a fractional-order differential operator. This element enables a more accurate modeling of energy dissipation phenomena in materials with memory effects.

Two fundamental generalized-modified fractional models were constructed:

- the **Kelvin–Voigt model of the fractional type** ($K_\alpha = H/N_\alpha$), which exhibits the property of **subsequent elasticity**, and
- the **Maxwell model of the fractional type** ($M_\alpha = H - N_\alpha$), which demonstrates **normal stress relaxation**.

For both models, analytical expressions were derived using Laplace transforms, and the resulting surfaces of axial dilatation and stress relaxation were presented as functions of time and the fractional differentiation exponent $\alpha \in (0, 1)$. These models serve as foundational elements for constructing more complex rheological systems.

Furthermore, a series of **generalized-modified fractional rheological models** was proposed, including fractional versions of the Bingham, Lethersich, Jeffrey, and Burgers models. Each model is defined by a specific structural formula and corresponding fractional-order constitutive relations, offering a unified framework for describing the mechanical behavior of idealized materials with complex internal structure and memory.

The results presented in this paper provide a theoretical basis for further development of **fractional rheological dynamic systems**, including oscillatory and creep-type systems with internal degrees of freedom. These models are particularly relevant for describing the behavior of advanced materials such as polymers, biomaterials, and piezoelectric composites under dynamic loading conditions.

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