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SCHOOL OF ELASTICITY AND FRACTURE MECHANICS AT THE FACULTY OF MECHANICAL ENGINEERING, NIŠ: RESEARCH AND GLOBAL COLLABORATION

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Abstract. *This paper presents a part of the scientific results in elasticity theory and fracture mechanics achieved through projects at the Department of Mechanics, Faculty of Mechanical Engineering, Niš. It recalls the doctoral dissertation of Prof. Dr Danilo Rašković on rectangular rod torsion, partly included in Jakov Hlitčijev's Theory of Elasticity. Under Rašković's guidance and through nine project cycles, numerous defended doctorates contributed to founding the Serbian School of Nonlinear Oscillations and Hybrid Systems Dynamics. Special focus is on parallel research in elasticity and fracture mechanics. Notable results include Dragan Jovanović's master's thesis on elasticity and photoelasticity and his doctoral work in fracture mechanics, as well as Ljubiša Perić's research on stress, deformation, and polarisation in piezoelectric ceramics. Both were mentored by Prof. Katica (Stevanović) Hedrih. International collaborations include Prof. Emanuel Gdoutos (University of Xanthi and Chicago), honorary doctorate recipient and guest editor of a Facta Universitatis special issue on fracture mechanics; Prof. Jerzy Pinder (Waterloo University); Prof. Stjepan Jecic (Zagreb); and Prof. Rastko Čukić (Belgrade). Key fracture mechanics results by Dragan Jovanović were recognised at the World Conference on Experimental Mechanics. Recent advances (2024–2025) include new fractional-order rheological models and dynamic systems by Katica Hedrih, Anđelka Herdrih, and Julijana Simonović. The paper also reviews the textbook-monograph Strength of Materials and Resistance of Structures by Dragan Jovanović and Julijana Simonović. References include works by Hlitčijev, Rašković, Hedrih, Jovanović, and Simonović, forming the knowledge base for generations of mechanical engineers. This paper emphasises the continuity of knowledge transfer and scientific ideas at the University of Niš since its founding in 1960.*

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1. INTRODUCTION

1.1. The first university professor of mechanics and machine science, appointed on January 28, 1876, by the decree of Rector St. Bošković at the Great School, founded by Dositej Obradović, was Academician Ljubomir Klerić (1844–1910) (see Fig. 1), one of the first ten academicians of the Serbian Royal Academy of Sciences. From that Great School, the University of Belgrade later developed, which is the parent university for all universities in Serbia. By education, Ljubomir Klerić was a mining engineer. He also wrote the first university textbook [1], "Classical Mechanics According to Weisbach" with a total of 1,317 pages in Serbian, published in three parts: 1880 (pp. 1–628), 1883 (pp. 629–1072), and 1889 (pp. 1073–1317), intended for students. He was a designer of precise and complex kinematic mechanisms and technical devices used in mining, (see Refs [2 -10]). In practice, Ljubomir Klerić (1844–1910) is considered the founder of the Serbian School of Mechanical Engineering and the Serbian School of Mechanics.

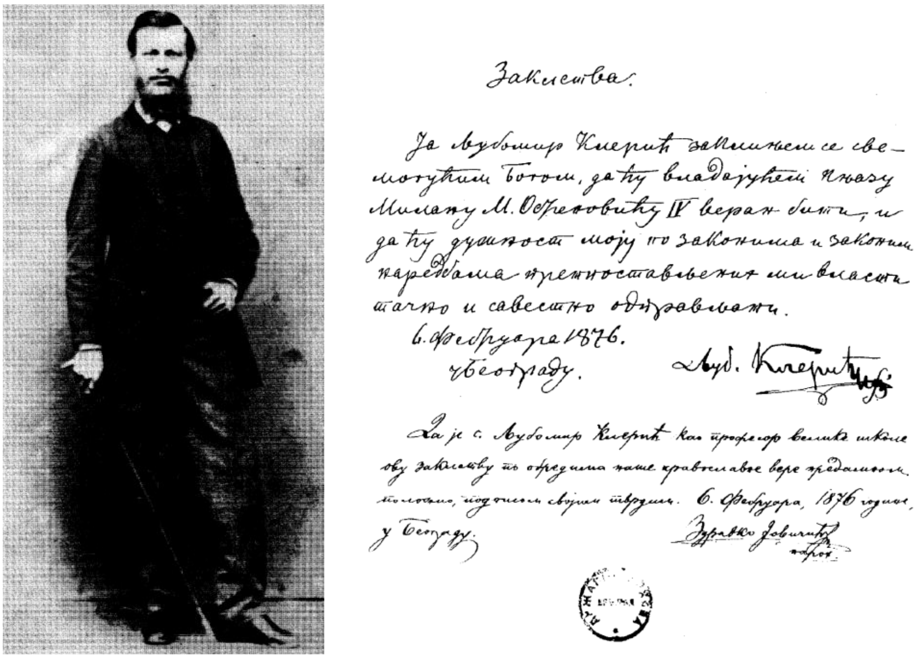


Fig. 1 The first university professor of mechanics and machine science, Academician Ljubomir Klerić (1844-1910) and Excerpt from the oath of office before the reigning Prince Milan Obrenović IV for the title of professor at the Great School, founded by Dositej Obradović (Archives of Serbia, University of Serbia, 1876/12)

1.2. At the beginning of the last century and the previous millennium, at the University of Belgrade, two doctoral dissertations on ball rolling were defended as the first in the field of mechanics (see Refs. [11, 12]). The topic of the first dissertation was *Rolling without Sliding of a Gyroscopic Ball on a Sphere*, defended in 1924 by Vasilij Demchenko (see Ref. [11]). The second dissertation, entitled *Rolling of a Heavy, Rigid Ball on an Elastic Surface* (see Ref. [10]), was defended in 1932 by Konstantin Voronec (see Ref. [12] and Fig. 1d, e), who later became the founder of the Department of Fluid Mechanics at the Faculty of Mechanical Engineering in Belgrade and the first Head of the Department of Mechanics at the Mathematical Institute of the Serbian Academy of Sciences and Arts. Both doctorates [11, 12] were defended before committees composed of the same members, despite being separated by a decade.

The composition of both commissions included prominent academics of the Serbian Royal Academy of Sciences: Anton Bilimovič (1879–1970), Mihailo Petrović (1868–1943), and Milutin Milanković (1879–1958) (see Refs. [13–19] and Fig. 2). Academician Anton Bilimovič (see Refs. [13, 14–19] and Fig. 2a) was a leading scientist in mechanics and one of the initiators of the foundation of the Mathematical Institute of the Serbian Academy of Sciences and Arts, as well as the creation of the journal *Publications de l'Institut Mathématique*, originally intended for mathematics and mechanics but later transformed into a purely mathematical journal.

Academician Mihailo Petrović (see Ref. [13] and Fig. 2c) was a mathematician and founder of the Serbian School of Mathematics, and the author of the famous and fundamental work *Elements of Mathematical Phenomenon*, published only in Serbian and still insufficiently known internationally. He held ten patents in mechanical engineering and was one of three doctoral students of the renowned scientist Jules Henri Poincaré. He also invented the hydro-integrator for solving Riccati's differential equation, an innovation considered the forerunner of the analog computer.

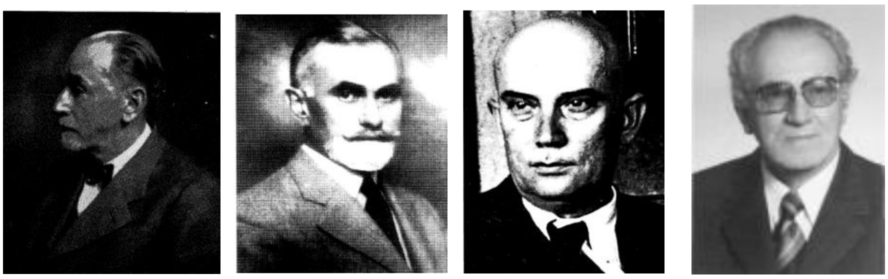
Academician Milutin Milanković (see Refs. [13, 14–19] and Fig. 2b) was one of the three world-renowned giants of Serbian science, alongside Nikola Tesla and Mihailo Pupin. His most significant global contribution is the *Canon of Solar Insolation*, while his *Celestial Mechanics*, written in Serbian, remains a cornerstone of Serbian scientific education.

a) b) c) d) e)

Fig. 2 a) Anton Bilimovič (Žitomir, 8/20 July 1879 – Belgrade, 17 September 1970)
b) Milutin Milanković (Dalj, 16/28 May 1879 – Belgrade, 12 December 1958)
c) Mihailo "Alas" Petrović (Belgrade, 24 April/6 May 1868 – Belgrade, 8 June 1943)
d, e) Konstantin Voronjec (Kiev, 17 January 1902 – Belgrade, 19 October 1974)

1.3. Professor Dr. Dipl. Mathematician Danilo P. Rašković (1910–1985) (see Refs. [20–27] and Fig. 3d) transferred from the Faculty of Mechanical Engineering in Belgrade to the Mechanical Engineering Department of the Technical Faculty, founded in 1960, in Niš in January 1964. He established the Department of Mechanics and taught all mechanics courses (Mechanics 1 – Statics, Mechanics 2 – Kinematics, Mechanics 3 – Dynamics, Mechanics 4 – Theory of Elasticity and Theory of Oscillations; see Refs. [21–27]) both at the Mechanical Engineering Department and the Electronics Department of the Technical Faculty in Niš, and later at the Faculty of Mechanical Engineering in Niš (FMEN). By primary education, Rašković was a mechanical engineer and a graduate mathematician, having previously completed military academy and high school.

Danilo P. Rašković defended his doctoral dissertation entitled *Tangential Stresses of a Normal Profile Beam* on June 21, 1944 (see Ref. [20]), before a commission consisting of full professors of the Faculty of Technical Sciences in Belgrade: Dr. Ivan Arnovljević, Jakov Hlitčijev, and Dr. Radivoj Kašanin (see Refs. [13, 28] and Fig. 3 a), b), c). The commission unanimously awarded both the thesis and the defense an excellent grade. Academician and Professor Radivoj Kašanin, mathematician (see Fig. 3c), particularly emphasized Rašković's original mathematical contribution in solving a differential equation with predetermined initial and boundary conditions, presented in this doctoral dissertation.



a) b) c) d)

Fig. 3 a) Academician Ivan Arnovljević (1896-1951); b) Academician Jakov Matvejevič Hlitčijev (1886−1963); c) Academician Radivoj Kašanin (1892-1989) members of Commission for doctoral dissertation defense and d) Professor Dr Ing Dipl Math Danilo P. Rašković (1910-1985)

Later, in his university textbook on the Theory of Elasticity, Academician Jakov Matvejevič Hlitčiev, in the chapter on torsion of rods with rectangular cross-sections, included and cited scientific results from Danilo Rašković's doctoral dissertation. Here we highlight the graphs of lines of constant tangential stresses in a rectangular cross-section beam under torsion. Hlitčiev emphasized this at the defense of Rašković's dissertation, assessing that these results would be included in many university textbooks on the Theory of Elasticity worldwide. Professor Radivoj Kašanin, during the defense, particularly noted that the problem Rašković addressed was reduced to a first-order differential equation that cannot be solved in closed form. He pointed out that the nature of its solutions could be examined analytically and then, through numerical integration, a solution obtained that satisfies the given initial and boundary conditions. Rašković successfully accomplished both tasks (see Refs. [13, 28]).

We also emphasize Rašković's extensive collection of university textbooks covering all areas of mechanics: Statics, Kinematics, Dynamics, Theory of Oscillations, Theory of Elasticity, Analytical Mechanics, Resistance of Materials, Tables of Resistance of Materials, Matrices, and Kinematics of Mechanisms, accompanied by problem collections. These were published by *Naučna knjiga* (see Refs. [21, 22, 24, 25, 27]), *Zavod za izdavanje udžbenika* (see Ref. [23]), and *Građevinska knjiga* (see Ref. [26]). Each textbook is comprehensive and includes mathematical appendices. Notably, in the *Naučna knjiga* edition alone, 120,000 copies of his textbooks were printed—an unsurpassed record in technical sciences. At that time, until 1986, books were typeset manually using lead letters and mathematical symbols tied with string, whereas today, preparation is done digitally in PDF format on computers. These numerous textbooks, with unified notation, were highly useful, readable, and fully adapted for students to easily study and acquire knowledge across the entire mechanics' cycle. They provided an excellent knowledge base for all students of technical sciences.

As Head of the Department of Mechanics, Professor Danilo P. Rašković identified talented students during the teaching process and guided them toward scientific research. He recognized the talent of mechanical engineering student Katica Stevanović and mentored her diploma thesis on *Nonlinear Oscillations and Applications in Automatic Control Systems*. She defended this thesis before the start of her final year and was awarded the title of "Best Graduate Student of the 1966/67 Academic Year at the University of Niš." Following this, she spent 11 months at the Institute of Mathematics in Kiev, studying Asymptotic Methods of Nonlinear Mechanics under the mentorship of world-renowned scientist Yuri Alekseevich Mitropolski, where she passed the candidate's minimum in *Theoretical and Mathematical Physics*.

At the suggestion of Professor Rašković, magister's science studies were established in March 1970 at the Faculty of Technical Sciences in Niš, which Katica Stevanović, the first in her generation, completed in 1972. Upon her return from Kiev, under Rašković's mentorship, she successfully defended her magister's science thesis in 1973 and her doctoral dissertation in 1975 (see Refs. [29, 30]). The commission was chaired by Academician Tatomir P. Anđelić (see Refs. [14–19]), with members including Professor Jurij Korobov, an expert in automatic control, and Professor Veljko Vujičić.

With this first diploma thesis, magister's science thesis, and doctoral dissertation in nonlinear oscillations, Professor Rašković directed the Department of Mechanics toward research in this field. After his departure in late 1974, Elastodynamics and Strength of Materials were taught by Dr. Katica Stevanović (Hedrih) (see Refs. [31, 32]), while Statics and Mechanics (Kinematics and Dynamics) were first taught by Professor Milutin Marjanović and, after his departure in 1976, by Dr. Dušan Stokić.

Katica (Stevanović) Hedrih led magister's science studies and, through nine project cycles (see Appendix I – References II, List of Projects 1.P–10.P and Refs. [33, 34]), supervised numerous magister's science theses and doctoral dissertations in nonlinear oscillations and dynamics of hybrid systems of complex structures. Through these projects and defended works, the Serbian School of Nonlinear Oscillations and Dynamics of Hybrid Systems of Complex Structures was established.

After 2009, magister's science studies and the requirement to obtain the title of Magister of Science before pursuing a doctorate were abolished, and doctoral studies were introduced, requiring completion and defense of a doctoral dissertation. Today, the Department of Mechanics continues this tradition, mentoring new doctoral candidates and advancing the Serbian School of Nonlinear Oscillations and Dynamics of Hybrid Systems of Complex Structures.

Parallel to these researches in the field of nonlinear oscillations, and also through nine project cycles (see Appendix I – References II, List of Projects 1.P–10.P and Refs. [33, 34]), research in the field of solid mechanics (Theory of Elasticity, Fracture Mechanics, and Mechanics of Materials with Coupled Fields) was carried out. From these studies, five magister's science theses and two exceptional doctoral dissertations were written and defended (see Refs. [35–39]). These doctoral dissertations practically laid the foundations of the School of Solid Mechanics (Theory of Elasticity, Fracture Mechanics, and Mechanics of Materials with Coupled Fields) at the Department of Mechanics of the FMEN.

The most prominent scientific results in this field are contained in the magister's science theses and doctoral dissertations of Dragan Jovanović (see Refs. [35, 36] and Fig. 4a) and Ljubiša Perić (see Refs. [38, 39] and Fig. 4b). Therefore, we present the main scientific methods and results from these magister's science theses and doctoral dissertations.

a)

b)

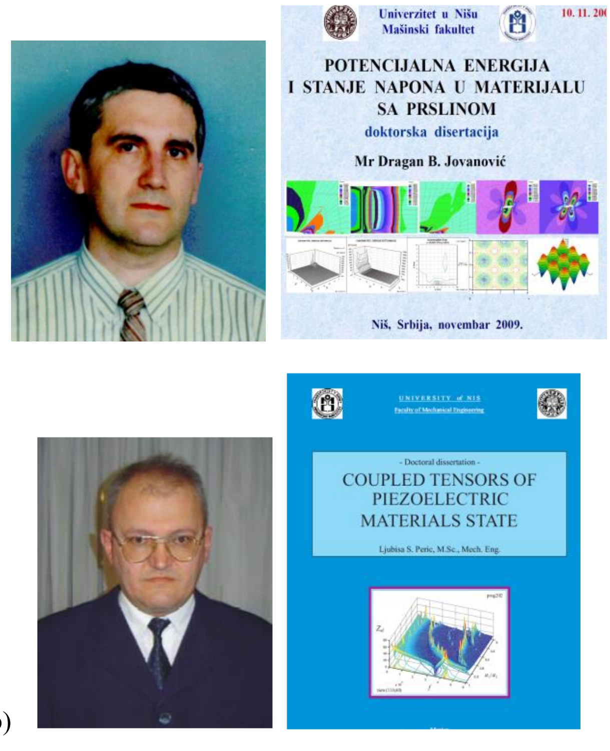


Fig. 4 a) Professor Dr Dragan B. Jovanović and his doctoral thesis [36]; b) Dr Ljubiša S. Perić, industrial partner and his doctoral thesis [39]

Dragan Jovanović achieved a number of original scientific results using experimental methods (see Refs. [35, 36, 40–51]). The Department of Mechanics and Jovanović himself established fruitful cooperation with Professors Stjepan Ječić (see Refs. [40, 41]), Head of the Laboratory for Photoelasticity at the Faculty of Mechanical Engineering in Zagreb; Professor Rastko Čukić (see Ref. [52]), Head of the Laboratory for Photoelasticity at the Faculty of Mechanical Engineering in Belgrade; and Professor Jerzy Pinder (see Refs. [49–51]), Head of the Laboratory for Experimental Mechanics and Optical Methods at the University of Waterloo in Canada. In these laboratories, he conducted experiments that produced new scientific results incorporated into his magister's science thesis and doctoral dissertation.

Over a long period, the Department of Mechanics also cooperated with a group of prominent scientists from Greece. From the National Technical University of Athens (NTUA), collaboration was realized with Academician Antoni Kounadis and Professor John Katsikadelis, as well as with Emmanuel Gdoutos (Fig. 5) from the Democritus University of Xanthi and the University of Chicago. All three delivered invited lectures at the Seminar on Theoretical and Applied Mechanics organized by our department and participated in the Congress of Mechanics in Serbia. They also served as members of editorial boards for certain series of the *Facta Universitatis* journal of the University of Niš. Later, all three distinguished scientists received the title *Doctor Honoris Causa* (Honorary Doctor of Science) from the University of Niš (see Fig. 5)

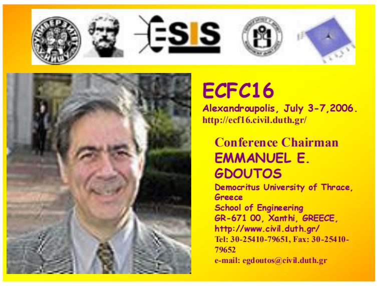


Fig. 5 Photo from the promotion of Professor Emanuel Gdoutos as a Doctor Honoris Causa - honorary Doctor of Science in 2013 at the University of Niš.

Professor Emmanuel Gdoutos, as Guest Editor, prepared a special issue of the journal *Facta Universitatis Series: Mechanics, Automatic Control and Robotics* dedicated to fracture mechanics. Dragan Jovanović and Katica Hedrih published their single author works in this special issue, upon invitation by the Guest Editor. Professor Gdoutos also organized a major European Conference on Fracture Mechanics in Alexandroupolis (see Fig. 6), where he invited Katica (Stevanović) Hedrih to organize a mini symposium, which she successfully implemented together with a larger group of researchers from the project she led. She also delivered an invited scientific lecture at the NTUA and participated by invitation in the Congresses of the Hellenic Society of Mechanics in Patras, Ioannina, and Xanthi.

Fig. 6 Greece and Serbian participants at Seminar of Theoretical and applied mechanics at Faculty of Mechanical Engineering University of Niš and participants of Mini-Symposia at ECFC16 Alexandroupolis 2006, Greece

2. ANALYSIS OF STRESS AND STRAIN STATES IN LOADED PLATES WITH APPLICATION TO AN ELLIPTICAL ANNULAR PLATE

The master's thesis [35] entitled *"Analysis of the Stress State and Strain State of Plane-Stressed Plates with Application to an Elliptical Annular Plate"* by Dragan Jovanović investigates the stress and strain state of a plane-stressed plate with two contours, using the example of an elliptical annular plate. The author analyzed the stress and strain state under different types of loading: concentrated forces and continuously distributed linear loads along the outer and inner elliptical contours. For his research, Jovanović applied the Kolosov–Muskhelishvili analytical method based on complex variable functions (see [53, 54]), as well as the numerical finite element method. All results obtained analytically and numerically were experimentally verified using the optical photoelastic method (see [55–61]). The experiments were conducted through optical photoelastic analysis of stress and strain states in two specialized laboratories: one at the Faculty of Mechanical Engineering in Zagreb, led by Professor Stepan Jecić (see [40, 41]), and the other at the Faculty of Mechanical Engineering in Belgrade, under the leadership of Professor Rastko Čukić (see [52]). Both professors extended their hospitality to the young researcher, then an associate of the FMEN, who demonstrated exceptional talent and broad mathematical and technical knowledge.

The scientific results obtained through these three methods remain relevant even today, decades later, and represent a significant contribution not only to the author but also to the Department of Mechanics at the FMEN. Most of these results have not yet been published, making it important to preserve and disseminate them in a high-quality scientific monograph as a valuable source of knowledge for future generations of doctoral students and young researchers. By applying the analytical method of complex variable functions, the components of the stress and strain tensors, as well as the displacement vector components of points on the elliptical annular plate, were determined. It is worth emphasizing that this section contains a complete analytical explanation of the Kolosov–Muskhelishvili method, which remains relevant today, especially since contemporary researchers tend to focus on the finite element method, while analytical approaches are unjustifiably falling into neglect.

Experimentally, using an optical photoelastic device for various concentrated and linearly distributed loads along elliptical contours, the author determined and plotted isoclines and isochrones on the surface of the elliptical annular plate, applying appropriate boundary conditions for this plate in the form of a doubly connected region of the elliptical annular surface.

$$\sigma_x + \sigma_y = 4\, Re\{F'(z)\}$$

$$\sigma_x - \sigma_y - 2i\tau_{xy} = 2\left\{z\overline{F}''(\overline{z}) + \overline{X}''(\overline{z})\right\}$$

In the previous expressions σ_x and σ_y are normal stresses, τ_{xy} is the shear stress of the plane stress state in the stressed plate. $F(z)$ is the analytic function of the complex variable, while $\overline{F}(z)$ is the conjugate function of the conjugate complex variable $\overline{z}$. $X(z)$ is an arbitrary function.

The analytical function $F(z)$ of the complex variable was assumed in the form of a power series in complex variables with unknown coefficients, which he determined from the boundary conditions set for the corresponding linear distributed surface loads on the contours of the elliptical-annular plate.

In his work, he also used the conformal mapping of the elliptical-annular plate into a circular-annular plate, which simplified the task of finding appropriate solutions for the component stresses, deformations and displacements of points of the elliptical-annular plate. Here we list only some analytical-mathematical expressions:

The Airie's stress function is a function of two analytical functions of the complex variable, and their conjugate complex expressions were used in the following form:

$$\varphi(z) = \frac{1}{2}\left[\overline{z}F(z) + z\overline{F}(\overline{z}) + X(\overline{z}) + \overline{X}(\overline{z})\right]$$

In which the functions $X(z)$ and $X(z)$ are arbitrary functions.

The coordinates of the displacement vectors of the points of the elliptical annular plate were expressed through these functions in the form:

$$u = \frac{1}{E}\left[4P - (1+\mu)\frac{\partial\varphi}{\partial x} + f_1(x)\right] \text{ and } v = \frac{1}{E}\left[4Q - (1+\mu)\frac{\partial\varphi}{\partial y} + f_2(y)\right]$$

In which u and v are the coordinates of the displacement vectors of the plate points in deformation and stress state, $f_1(x)$ and $f_2(y)$ arbitrary functions of complex variable z.

Essentially, as a single complex expression for the component u and v of displacement vector of a point on the plate in complex form, it was written:

$$u = \frac{1}{E}\left[4P - (1+\mu)\frac{\partial\varphi}{\partial x}\right] \; and \; v = \frac{1}{E}\left[4Q - (1+\mu)\frac{\partial\varphi}{\partial y}\right]$$

or in the following complex form

$$u + iv = \frac{1}{E}\left[4(P+iQ) - (1+\mu)\left(\frac{\partial\varphi}{\partial x} + i\frac{\partial\varphi}{\partial y}\right) + \left(f_1(x) + if_2(y)\right)\right]$$

or in the following complex form

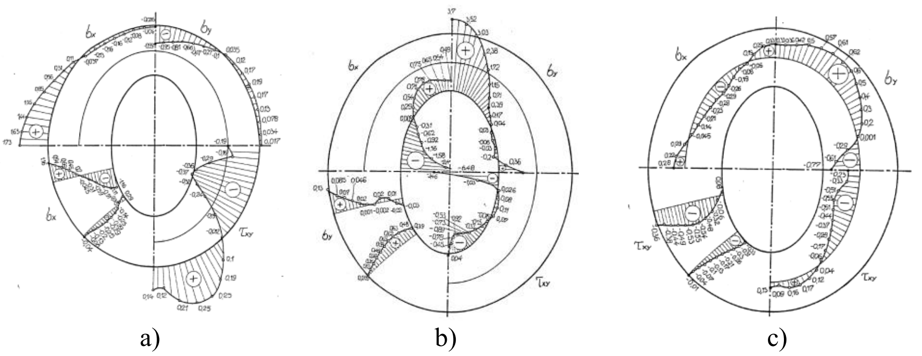
$$u + iv = \frac{3-\mu}{E}F(z) - \frac{1+\mu}{E}\left[z\overline{F'}(\overline{z}) + \overline{X'}\left(\overline{\overline{z}}\right)\right]$$

To engage readers in new research that applies analytical methods-especially in an era where technical science researchers are increasingly directed toward various commercial software. Ref. [35] provides detailed information on the name and application of this method for analyzing the stress state, strain state, and point displacements of an elliptical annular plate under different loads along its elliptical contour.

It is important to note that the master's thesis employs the method of conformal mapping, transforming an elliptical annular plate into a circular annular plate, along with mapping the loads applied along the elliptical contour. This approach significantly simplifies the problem. The method of conformal mapping, which converts a domain with complex contours into one with simpler contours, has proven highly effective in reducing the complexity of such problems. Although the mathematical representation of the mapped load along the circular contour becomes more intricate, the contour transformation itself is much simpler and enables a less complicated solution. The thesis includes several examples of conformal mapping of external and internal contour loads into equivalent circular contour loads.

In this review paper, we do not aim to provide a detailed explanation of the methods applied in the original thesis. Rather, we will present a selection of scientific results, illustrated through Figs. 7–13. The first series of figures (Figs. 7-9) depicts the stress component distribution for the case of concentrated forces acting on the inner elliptical contour in the direction of the minor axis of the ellipse.

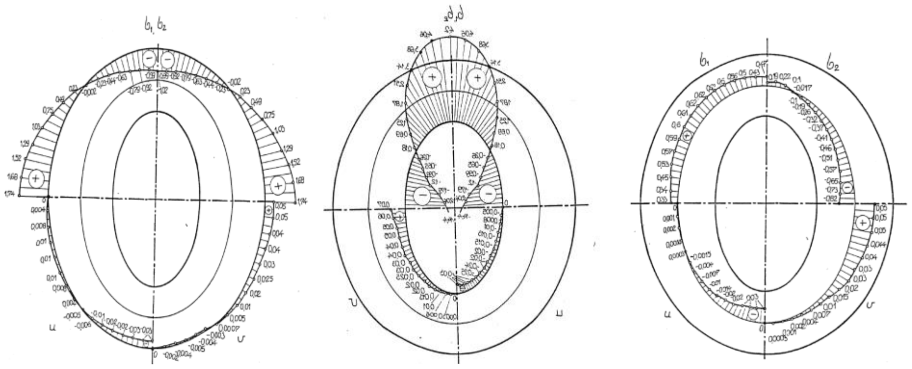
This review highlights only a small portion of the scientific results from this significant master's thesis, including stress lines of the stress tensor components, as well as isochrone and isocline lines. It should be emphasized that, by applying the Kolosov–Muskhelishvili method, the author introduced an analytical function of a complex variable and used it to express the relationships among the tensor components in the plane stress state.



a) b) c)

Fig. 7 Normal stress diagrams σ_x and σ_y and shear stress τ_{xy} at points of the outer (a) and inner elliptical contour (b), as well as at points of the middle elliptical section (c) of an elliptical annular plate loaded with concentrated forces on the inner contour.

The first case of loading an elliptical-annular plate is, by opposite concentrated forces, at points, on the internal elliptical boundary contour, in the direction of the minor semi-axes of the elliptical contour. Figs. 7 a), b) and c) show diagrams of normal stresses σ_x and σ_y, as well as shear stress τ_{xy} at points of the outer (a) and inner elliptical contour (b), as well as at points of the middle elliptical section (c) of an elliptical annular plate loaded with concentrated forces on the inner contour, obtained by the numerical Finite Element Method FEM . The same figures a, b and c show diagrams of component stresses of that plane stress at points of characteristic cross-sections of the plate with the corresponding labels in the figures. The diagrams are compiled based on numerical data obtained by the FEM. From these diagrams, it is observed that there is a greater increase in the gradient of normal stresses σ_x and σ_y, as well as shear stress τ_{xy} in the vicinity of the loading place of concentrated forces acting at points on the inner elliptical contour.

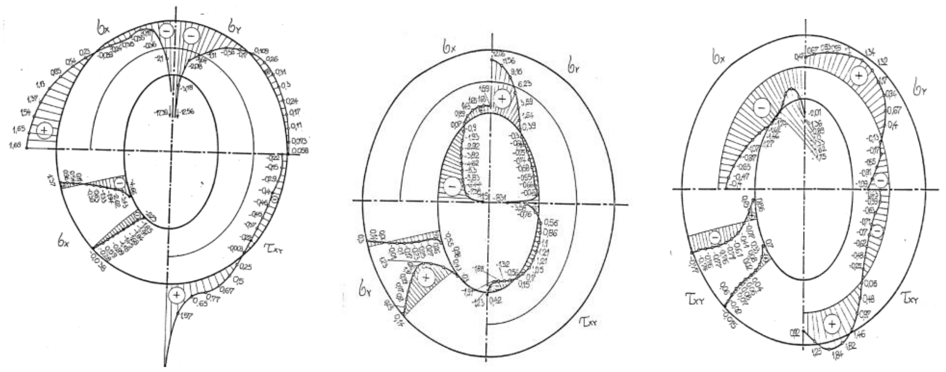


a) b) c)

Fig. 8 Diagrams of the principal normal stresses σ_1 and σ_2 , as well as the component displacements u and v at points a) of the outer and b) of the inner elliptical contour, as well as c) of the central conformal elliptical section of an elliptical annular plate loaded with concentrated forces with attack points on the inner contour in the direction of the minor semi-axes of the elliptical contour.

Fig. 8 shows the principal normal stresses σ_1 and σ_2, as well as the component displacements u and v at points a) on the outer; b) on the inner elliptical contour, as well as c) of the central conformal elliptical section of an elliptical annular plate loaded with concentrated forces with attack points on the inner contour in the direction of the minor semi-axes of the elliptical contour.

The second loading case of elliptical annular plate is when two opposite concentrated forces act at opposite points of attack on the outer elliptical contour, in the directions of the longer semi-axes of the elliptical contour, of the elliptical annular plate, stressing it in compression. Here we show only selected diagrams shown in Fig. 9 a), b) and c).



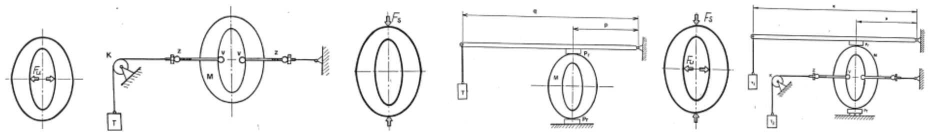
a) b) c)

Fig. 9 Diagrams of normal stresses σ_x and σ_y, as well as shear stress τ_{xy}: a) at points of the outer elliptical contour, as well as normal stress σ_x in the characteristic cross-section; b) at points of the inner elliptical contour, as well as normal stress σ_x in the characteristic cross-section; c) at points of the inner middle confocal elliptical cross-section, as well as normal stress σ_x in the characteristic cross-section; for the case of a plane state of stress of an elliptical annular plate by two opposite concentrated forces with attack points on the outer contour, in the directions of the longer semi-axes of the elliptical contour

Fig. 9 a) shows diagrams of normal stresses σ_x and σ_y, as well as shear stress τ_{xy} at characteristic points of the outer elliptical contour, as well as normal stress σ_x in the characteristic cross-section shown in the figure, for the case of plane stress of an elliptical annular plate by two opposite concentrated forces, with the attack points on the outer contour in the directions of the longer semi-axes of the elliptical contour. The same type of diagrams for the points of the inner elliptical contour is shown in Fig. 9 b), whareas Fig 9 c) illustrates the same stress state at points along the inner middle confocal elliptical cross-section for the same loading condition.

The third method, which was used to investigate the stress state of an elliptically annular plate in plane stress, is the photoelastic method. Models made of optically active material and a polariscope were used, and the polarization of monochromatic light and multichromatic light in the device - a polariscope - was used. The paper presents isocline and isochrome maps obtained on a model made of optically active material, and by obtaining results on an analyzer for different types of loads and compiling isocline and isochrame maps, the stress state is shown in Figs. 11, 12 and 13.

Figs. 10 a), b) and c) show the sketches of the equipment for the plane loading of an elliptically annular plate with concentrated forces with attack points on the inner elliptical contour in the directions of the shorter semi-axes of the elliptical contours, on the outer contour in the directions of the longer semi-axes of the elliptically annular plate, as well as when the elliptically annular plate is loaded with two pairs of opposite concentrated forces, on the inner contour in the directions of the shorter semi-axes of the elliptical contours, as well as on the outer contour in the directions of the longer semi-axes of the elliptically annular plate.



a) b) c)

Fig. 10 Sketches of the device for plane loading of an elliptical annular plate with concentrated forces with attack points a) on the inner contour in the directions of the shorter semi-axes of the elliptical contours, b) on the outer contour in the directions of the longer semi-axes of the elliptical annular plate and c) on the inner contour in the directions of the shorter semi-axes of the elliptical contours, as well as on the outer contour in the directions of the longer semi-axes of the elliptical annular plate

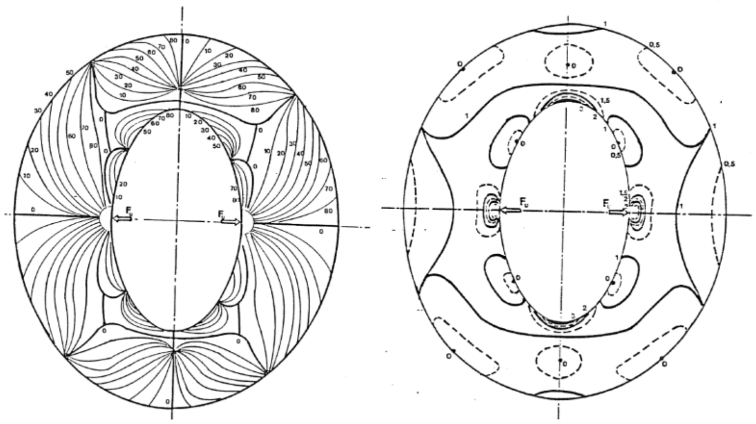
An isocline is defined as the locus of points forming a line—called an isocline—where the axes of the principal stress directions coincide in a plane-stressed plate. On a polariscope, for a given orientation of the optical axes of the polarizer and analyzer, a single line corresponding to one isocline becomes visible. Individual isoclines are then combined to create an isocline map. Three such maps, taken from D. Jovanović's master's thesis, are shown in Figs. 11 a), 12 a), and 13 a), corresponding to different loading conditions involving opposing concentrated forces acting on an elliptical annular plate.

An isochrone map represents the visual image of the stress state of a plane-loaded elliptical annular plate when a model made of optically active material is placed in a polariscope. When the optical axes of the polarizer and analyzer are crossed at right angles α and the model is illuminated, dark lines appear at all points where the light beam retardation is an integer multiple of the wavelength. Conversely, when the optical axes of the polarizer and analyzer are parallel, bright fields are observed.

Isochrones are the loci of points forming lines—called isochrones—observed on the model through the analyzer, where the difference between the principal stresses remains constant. The term "isochrones" is used because, when white light passes through the polarizer, a spectrum of colors appears on the model for different wavelengths (for details, see Refs. [35, 55–57, 60, 61]).

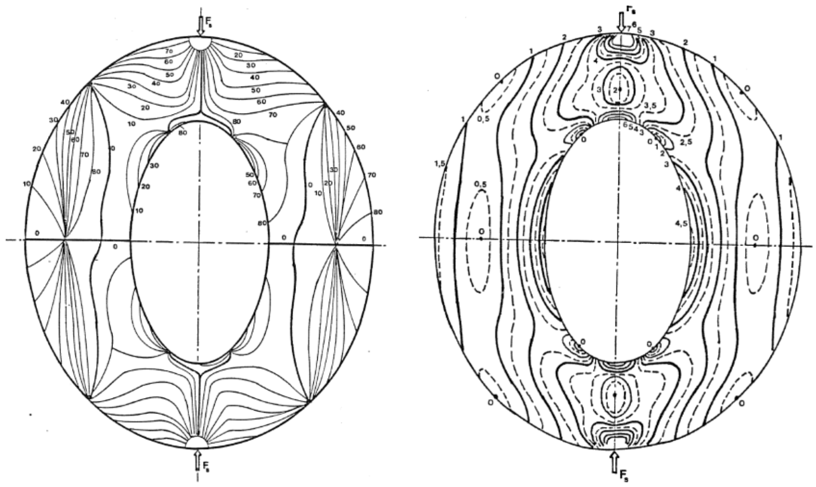
Three such isochrome maps reconstructed on a model of a loaded elliptical annular plate, Figs 10, made of optically active material, from D. B. Jovanović's magister's science thesis, are presented in Figs 11 b), 12 b) and 13 b) for different plane loadings with opposing concentrated forces on the elliptical annular plate. And to clarify, isochromes, on the isochrome map on the model observed in the analyzer field, are the geometric locations of points connected in lines (dark and light fields) of the same principal stress differences $\sigma_1 - \sigma_2$ of the order of m or $(2m+1)/2$ in the following form:

$$(\sigma_1 - \sigma_2) = \begin{cases} \dfrac{S}{d} m & \alpha = 0 \\ \dfrac{S}{d} \dfrac{2m+1}{2} & \alpha = \dfrac{\pi}{2} \end{cases}$$



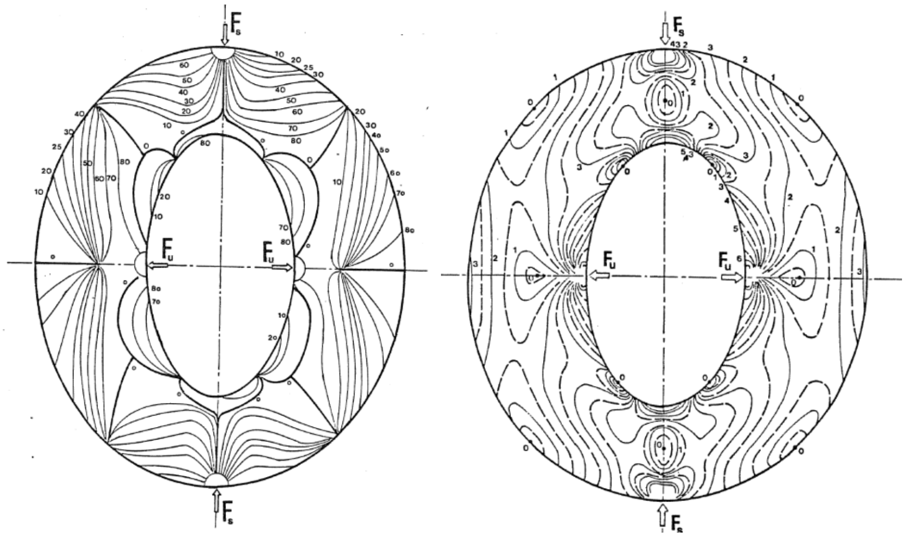
a) b)

Fig. 11 Results of photoelastic analysis, in the analyzer, of a stressed elliptical annular model loaded by a pair of opposite concentrated forces, with attack points on the inner elliptical contour in the directions of the shorter semiaxes of the elliptical confocal contours: a) an isocline map and b) an isochrone map



a) b)

Fig. 12 Results of photoelastic analysis, in the analyzer, of a stressed elliptical annular model loaded with a pair of opposite concentrated forces with attack points on the outer elliptical contour in the directions of the longer semi-axes of the elliptical confocal contours: a) isocline map and b) isochrone map



a) b)

Fig. 13 Results of the summary photoelastic analysis of a stressed elliptical annular model, loaded with two pairs of opposite concentrated forces with attack points on the inner and outer elliptical, confocal contours, with one pair acting in the directions of the longer and the other pair in the directions of the shorter semi-axes of the confocal elliptical contours: a) isocline map and b) isochrone map

3. POTENTIAL ENERGY AND STRESS STATE IN MATERIAL WITH CRACK

The doctoral dissertation [36] entitled *"Potential Energy and Stress State in Cracked Materials"* by D. B. Jovanović was successfully completed and defended on November 10, 2009, at the Department of Mechanics, FMEN. The dissertation comprises both theoretical and experimental research. A portion of the theoretical results was published in [47], in a special issue of the university journal *Facta Universitatis, Series Mechanics, Automatic Control and Robotics*, University of Niš. This special issue was edited by invited Guest Editor Professor Dr. Emmanuel Gdoutos from Democritus University of Thrace, Greece, and the University of Chicago, USA—a distinguished world authority in fracture and damage mechanics.

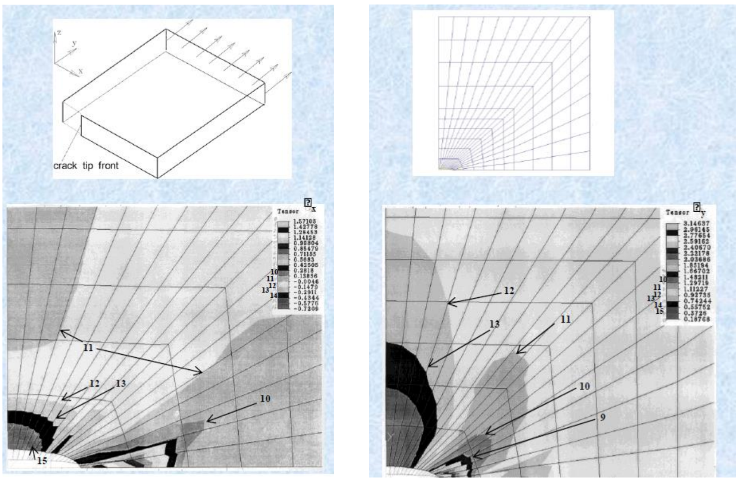
The experimental part of the research was conducted in the Laboratory for Experimental Mechanics at the University of Waterloo, Canada, under the supervision of the renowned Professor Jerzy T. Pindera. These results reflect collaboration among three internationally recognized institutions: FMEN, the University of Waterloo, and Democritus University of Thrace. The central figure in this collaboration was D. B. Jovanović, whose skill and enthusiasm as a talented researcher were supported by Professors Jirji Pindera, Emmanuel Gdoutos, and Katica (Stevanović) Hedrich. Later, Professor Emmanuel Gdoutos was awarded the title of Doctor Honoris Causa by the University of Niš. Similarly, through cooperation between the Chair of Mechanics at FMEN and Greek scientists from the National Technical University of Athens, Professors Antoni Kounadis and John

Katsikadelis also received the honorary title of Doctor Honoris Causa from the University of Niš.

In this review, we have chosen to present selected theoretical and experimental results from Professor Dr. D. B. Jovanović's doctoral dissertation. For comprehensive details, readers are referred to the original dissertation [36]. Portions of this research, co-authored with Professor Pindera, were presented at the World Congress of Experimental Mechanics and published in Refs. [49–51].

The doctoral dissertation of Dr. D. B. Jovanović [36] is not only a work that merited the title of Doctor of Science in fracture mechanics but also represents a highly significant scientific contribution by the candidate and the Department of Mechanics at FMEN. This achievement stands as a testament to successful international collaboration and should remain a source of pride for future generations of the Department of Mechanics.

The following Figs. 14, 15 and 16 show selected experimental results of the researcher presented in this doctoral dissertation.

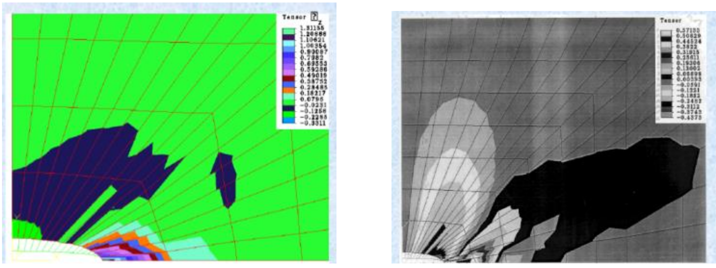


a) b)

Fig. 14 a) Normal stress σ_x distribution, for a plane cross-section and b) Normal stress σ_y distribution, for a plane cross-section z = 0, in detail, in a flat plate stressed in tension

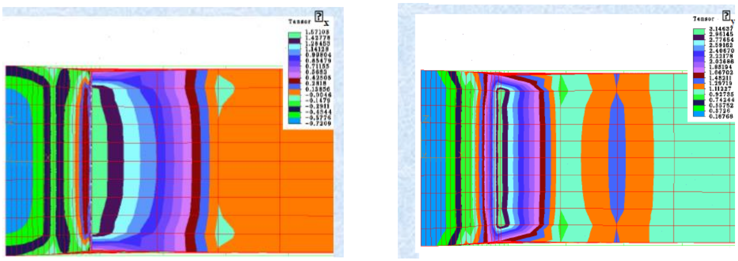
Figs. 17–21 present selected theoretical results from the doctoral dissertation [36] and from published references [45–48]. In the presentation of these results, a characteristic key statement was emphasized: *"Strain energy in the vicinity of an elliptical crack in a plate loaded with a normal tensile stress in the direction perpendicular to the crack plane…"*, which defines the focus of the experimental investigation.

In his paper [47], D. B. Jovanović presents some of the theoretical results later included in his doctoral dissertation. He writes: *"The theory of fracture mechanics has two main approaches to the problem of crack propagation: continuum mechanics and the atomic approach. These approaches are presented in the classical literature on fracture mechanics."*



a) b)

Fig. 15 a) Normal stress distribution in the z direction, for a section z = 0 and b) Tangential stress distribution, for a plane section z = 0, in a flat plate under tensile stress



a) b)

Fig. 16 a) Normal stress σ_x distribution, for a plane section y=0 and b) Normal stress σ_y distribution, for a plane section y = 0, in a flat plate under tensile stress

The published paper [47] focuses on the atomic perspective: *"The atomic approach considers cracks within a discrete model of material (atomic lattice). Solids may be represented as systems of discrete masses linked by interacting forces—interatomic forces or simple bonds, Fig. 19 a). Crack growth is influenced not only by mechanical loads but also by chemical, thermo-mechanical, electro-mechanical, acoustic, and other physical phenomena. The ability of the discrete model to explain crack healing, slow subcritical crack growth, sound generation during crack propagation, effects of chemical processes at the crack tip, nonlinearity of stress, strain, and energy distribution in the crack tip region, and the influence of temperature on crack propagation provides significant advantages to procedures based on discrete mass models (atomic lattices). Two intrinsic interatomic force functions are used to represent mechanical interactions between neighbouring atoms (discrete masses) in the lattice. Released potential energy, resulting from crack propagation through the lattice by breaking interatomic bonds, is analyzed. One-dimensional and two-dimensional lattice models are presented, along with relations for the total potential energy of selected lattice configurations."*

The key terms of this theoretical paper include: crack, mathematical form of localized energetic structure, discrete model of material, atomic lattice, interatomic force functions, potential energy of atomic bonds, total potential energy of lattice, activation energy, strain energy surfaces.

To illustrate and extend the presentation of these fundamental theoretical results, Figs. 17–21 are provided.

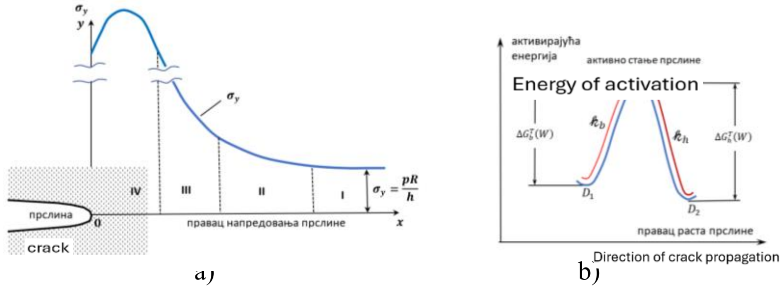
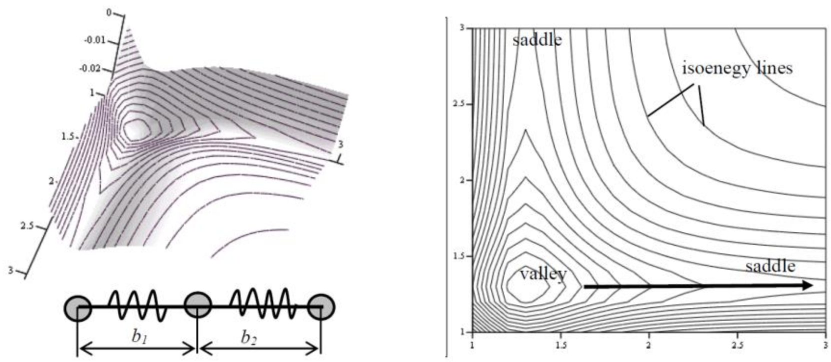


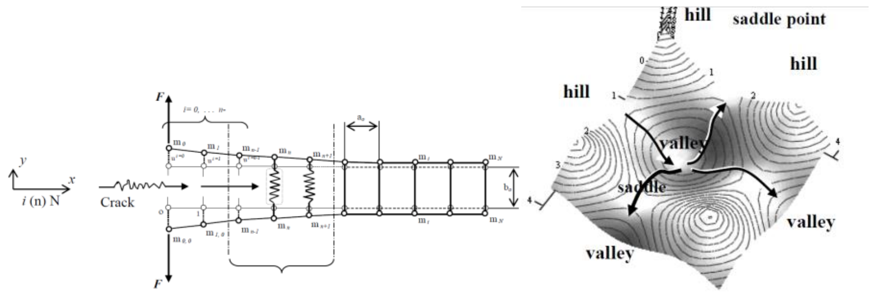
Fig. 17 a) Distribution of normal stress σ_y ahead of the crack tip, b) Sine wave approximation of energy activation function in atomic model of damaged plate



a) b)

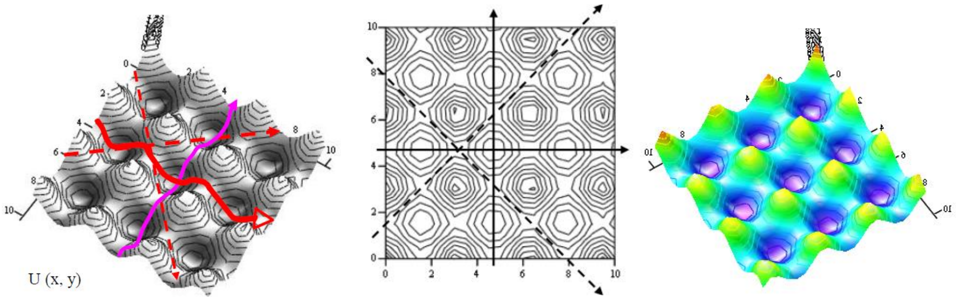
Fig. 18 a) Potential energy surface for a system of three atoms connected by interatomic interaction according to the assumed relation in [36]. b) Isoenergy lines projected onto the x-y plane, for a system of three atoms connected by interatomic interaction according to the assumed relation in the work [36]

Fig. 21 shows the distribution of the potential strain energy for selected directions z = 0 mm, z = 1 mm of the intersection y = 0 mm. Then, data files are merged into a single file with data on potential strain energy of the flat section defined by y = 0 mm. Then, D. B. Jovanović made a series of iterative steps, which reconstructed the surface describing the potential energy distribution in the plane y = 0 mm, as it is shown in Fig. 22.



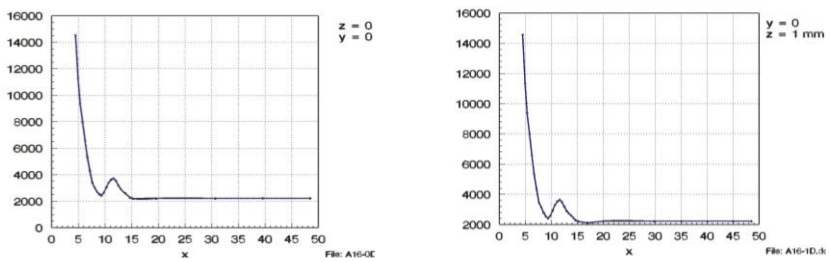
a) b)

Fig. 19 a) One-dimensional model of a network of atoms, and a crack that progresses through a chain of atoms [36]; b) Surface of potential energy for a two-dimensional *x-y* network in plate, and possible trajectories of crack progression through saddle points [36, 48]



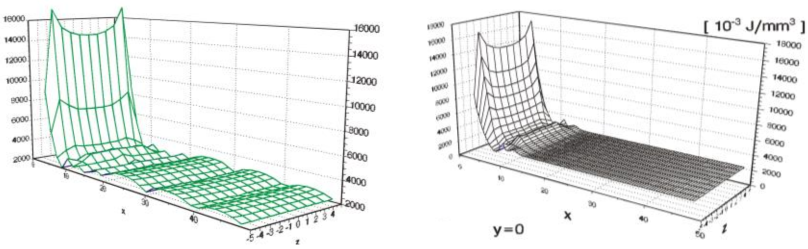
a) b) c)

Fig. 20 Potential energy surface for a two-dimensional x-y grid; a) and c) Potential energy surface for a two-dimensional x-y grid; b) Projection (x-y) of potential energy surface and possible directions of crack propagation in stressed plate [36]



a) b)

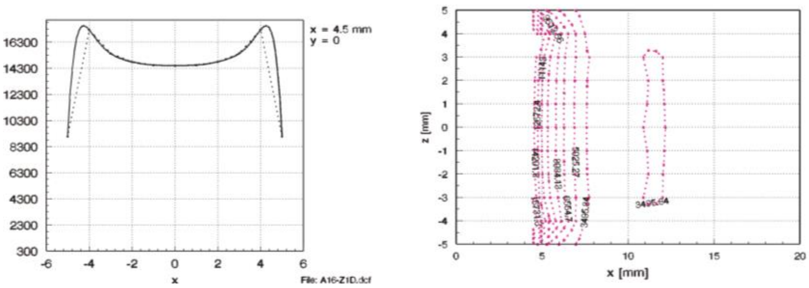
Fig. 21 Diagram of the specific strain energy in the direction determined by the coordinates: a) z = 0 mm and y = 0 mm; and b) z = 1 mm and y = 0 mm



a) b)

Fig. 22 Diagram of the surface of the specific strain energy of the flat section for coordinate y = 0 mm, in front of the crack's front line: a) the previous step of reconstruction and b) the final form of reconstruction.

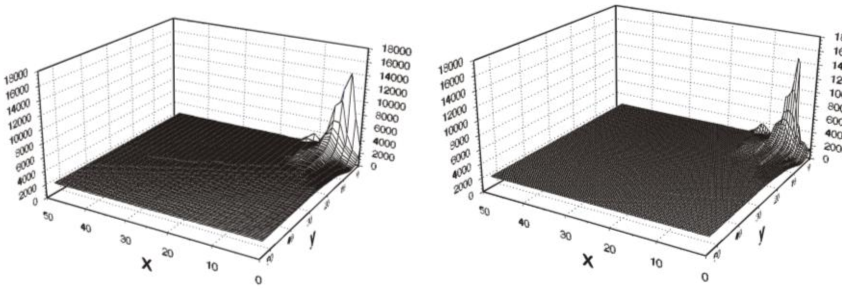
Fig. 23 a) shows a diagram of the potential strain energy in the direction x = 4.5 mm, and plane cross-section y = 0 mm. This diagram shows that minimum of potential strain energy ahead of the crack in the middle of the plate cross-section, and maximums near the outer surfaces of the plate. This fact suggests that the points on front line of crack, which are located near the outer surface of plate will get crack propagation, before the middle of the cross section of the plate. Fig. 23 b) shows the iso energy lines in the cross section at the z coordinates. These lines show the constant value of the potential energy of deformation, illustrating the distribution and concentration of potential strain energy, and hence point to places where it can be expected crack propagation. Fig. 23 shows a step in the iterative procedure.



a) b)

Fig. 23 a) Diagram of the specific strain energy in direction determined by the coordinates x = 4.5 mm and y = 0 mm; b) Diagram of the isoenergy lines in front of crack in plane section of plate, coordinate y = 0 mm

Fig. 24 illustrates the distribution of potential strain energy in the mid-plane, showing a peak value of $A'_{\mathrm{def}} = 14{,}517 \times 10^{-3}$ J/mm^3 located ahead of the crack front, similar to the diagram in Fig. 22 b). Additionally, a local maximum of strain energy is observed at a distance of 12 mm from the origin. This local peak results from a variation in the normal stress distribution law, which exhibits a local extreme at that point. However, the value of potential strain energy at this location is significantly lower than that at the crack tip.

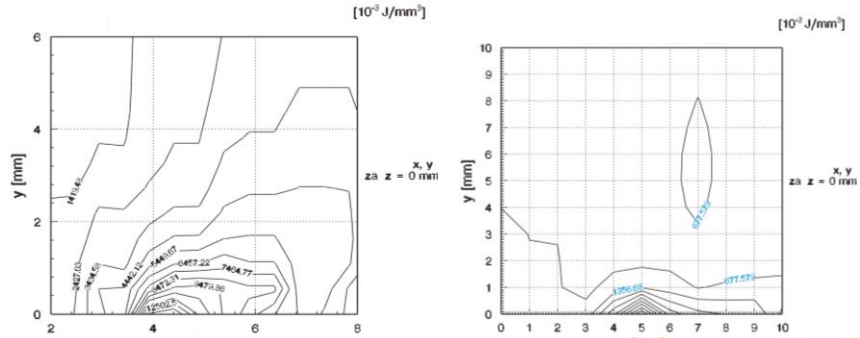


a) b)

Fig. 24 Diagram of the specific strain energy for flat section of plate, coordinate z = 0 mm: a) the previous step reconstruction and b) the final step of reconstruction

Fig. 25 presents the isoenergy lines in the mid-plane of the plates, revealing a pronounced concentration of strain energy ahead of the crack tip and a depression of the energy field above the crack in cases a) PR16 and b) PR22.

The results of applying the concept of reconstructing the spatial surface that describes the potential strain energy distribution in the vicinity of the crack are presented. Specifically, D. B. Jovanović analyzed two examples of cracks identified as PR-Z-16 and PR-P-22. The geometry of the cracks and plates was identical, and the same finite element mesh was used for stress calculations in both cases. The only difference between the examples was the loading method. In the PR-Z-16 case, the plate was loaded axially along the Y-axis, perpendicular to the crack plane, whereas in PR-P-22, the plate was subjected to a continuous load along the crack surface contour. Based on the reconstructed potential strain energy surface, isoenergy lines ahead of the crack tip were obtained for both cases. The analysis revealed that the stress distribution and the potential strain energy distribution were very similar in both examples, highlighting the dominant influence of the geometric shape of the crack contour.



a) b)

Fig. 25 Isoenergy lines in the vicinity of the crack tip, plane of section (x-y) for z = 0 mm (shown in enlarged detail): a) PR16 and b) PR22

At last, but not least, we wish to draw the reader's attention to textbook [61]. This textbook offers a comprehensive introduction to the principles of strength of materials and structural mechanics, specifically designed for mechanical engineering students. It covers fundamental concepts such as stress, strain, elasticity, and energy methods, along with practical applications in mechanical design. The book adopts an inductive approach, combining theoretical foundations with problem-solving techniques across 16 chapters. Topics include axial loading, shear, torsion, bending, buckling, and complex stress states, enriched with historical context and modern engineering practices.

Intended for undergraduate courses, it serves both as a learning resource and a reference for future engineers. Moreover, this book brings together, in a modern, systematic, and accessible way, the knowledge that the authors have built over decades—through their own education and extensive work with generations of young engineers. It represents a synthesis of fundamental principles from mechanics of materials, fracture mechanics, and the essential concepts underlying the finite element method (FEM). In this way, it provides readers with a comprehensive and practically applicable foundation for understanding and solving complex engineering problems.

4. COUPLED TENSORS OF PIEZOELECTRIC MATERIALS STATE

During the highly successful development of the Electronic Industry (EI) in Niš, led by Director Vladimir Jasić in 1980, the magisters of science in electronics, M.Sc. Miodrag Prokić and M.Sc. Dragan Šarković, worked enthusiastically on the development of the ultrasonic transducer program. At their invitation, Professor Dr. Katica (Stevanović) Hedrih, then Head of the Department of Mechanics and lecturer of *Elastodynamics* at FMEN, joined them as an external consultant at EI Niš. This collaboration inspired further research, and Assistant Aleksandar Filipovski successfully completed and defended his magister's thesis in 1995, entitled: *"Energijska analiza longitudinalnih oscilacija štapova promenljivog preseka (Energy Analysis of Longitudinal Oscillations of Rods with Variable Cross Sections)"* [in Serbian], Magister of Sciences Degree Thesis, FMEN, 1995 (see Ref. [37]). Under the supervision of K. Hedrih (Stevanović), several papers based on this research were published (Refs. [62–64]). In addition, joint work with the two magisters of science in electronics, M. Prokić and D. Šarković, resulted in publications presented at scientific conferences and in the *Proceedings of FMEN* (Refs. [65, 66]).

This group of magister's students from EI Niš and Professor Hedrih (Stevanović) from FMEN was later joined by Ljubiša Perić, director of one of the MIN plants in Niš. Under her mentorship, he successfully completed both his magister's thesis [38] and doctoral dissertation [39], focusing on the investigation of the coupled state of the mechanical stress tensor and the electrical state tensor in piezoelectric materials, earning his Doctor of Science degree in 2004.

Further cooperation developed with Professor Dr. Milan Radmanović and his then young associate Dragan Mančić from the Faculty of Electrical Engineering in Niš. In 2002, a significant scientific achievement followed when Dragan Mančić successfully defended his doctoral dissertation entitled *"Modeling of Powerful Ultrasonic Sandwich Converters"* under the mentorship of Professor Radmanović (see Refs. [67–70]). It is also noteworthy that much later, M.Sc. Dragan Šarković earned his Doctor of Science degree in physics in 2006 by defending his dissertation entitled *"Ultrasonic Sprayers"* under the mentorship of Professor Vukota Bulatović at the Physics Group of the Faculty of Natural Sciences and Mathematics in Kragujevac (see Ref. [71]). As a prerequisite for his doctoral defense, M.Sc. Dragan Šarković published a paper entitled *"An Auxiliary Size Distribution Model for Ultrasonically Produced Water Droplets"* in the prestigious journal *Experimental Thermal and Fluid Science*, co-authored by K. Hedrih (Stevanović) and Vukota Bulatović (see Ref. [65]).

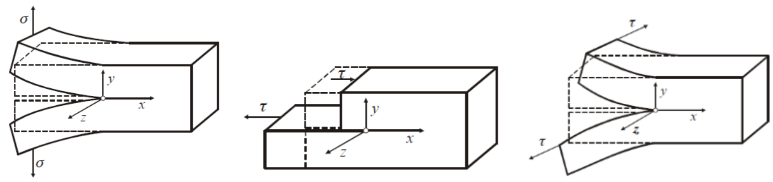
In the following section, we focus on selected details of the scientific results from Ljubiša Perić's doctoral dissertation (see Refs. [38, 39, 67, 68]), which he successfully completed and defended at FMEN under the supervision of Katica (Stevanović) Hedrih, addressing the coupled mechanical and electrical states of piezoelectric materials.

The electric polarization of a piezoelectric crystal, induced by an external mechanical load (tension or compression), is known as the piezoelectric effect. In this process, electrostatic charges appear on opposite surfaces of the piezoelectric medium, while electric polarization occurs within its interior. The piezoelectric effect arises from the mutual interaction of internally coupled fields—mechanical stress, deformation, and the electric field within the piezoelectric material. The results of investigations on electroelasticity in piezoelectric bodies are presented in the doctoral dissertation [39] and related papers [72, 73]. A selective review of methods for analyzing parameters of the coupled field state in piezoelectric bodies is provided, under the assumption that the material is of infinite dimensions and contains defects.

The focus of the study is the mechanical and electrical state at points near a semi-infinite crack, considered as a defect located within an idealized infinite piezoelectric material. The objective is to determine characteristic values and stress concentration factors that influence crack growth under combined mechanical and electrical loading. Using various methods, analytical solutions were obtained for the components of mechanical stress, dilatation, piezoelectric displacements, and electric potential in the vicinity of the crack for all three modes of crack deformation—addressing the spatial state. The method of complex variable functions was applied in polar-cylindrical coordinates. Furthermore, an analysis of the spatial stress and strain state was performed near the crack tip under conditions of material shear along the crack surface, both in and out of the reference plane, using Linear Elastic Fracture Mechanics (LEFM) and the linear theory of elasticity.

From the doctoral dissertation [39] by Dr. Ljubiša Perić, which he translated into English and made available online in PDF format, we present a selected excerpt to highlight this significant scientific work created two decades ago. Part of these scientific results—some developed in collaboration with his mentor and partly published in Refs. [67, 68, 72-74]—is summarized in the author's own words:

"A stress function and an electrical potential function were introduced, expressed in polar coordinates and represented by trigonometric series. The analysis revealed characteristic behavior of the coordinates of the mechanical stress tensor, the vector of piezoelectric displacement, and the tensor of relative mechanical deformations. Components of these tensors exhibit singular values at the tip of the semi-infinite crack. The vicinity of the crack tip, and the tip itself as a singular point of the stress and strain state, represents a zone of stress concentration and the most sensitive region with respect to subsequent fracture. Furthermore, a numerical analysis of the obtained analytical solutions was performed using the MATLAB software package. For a specific piezoelectric material (PZT-4), biparametric surfaces were generated to illustrate the influence of individual parameters on the crack growth velocity for all three modes of deformation."



a) b) c)

Fig. 26 a) **Mode I** crack deformation; b) **Mode II** crack deformation; c) **Mode III** crack deformation.

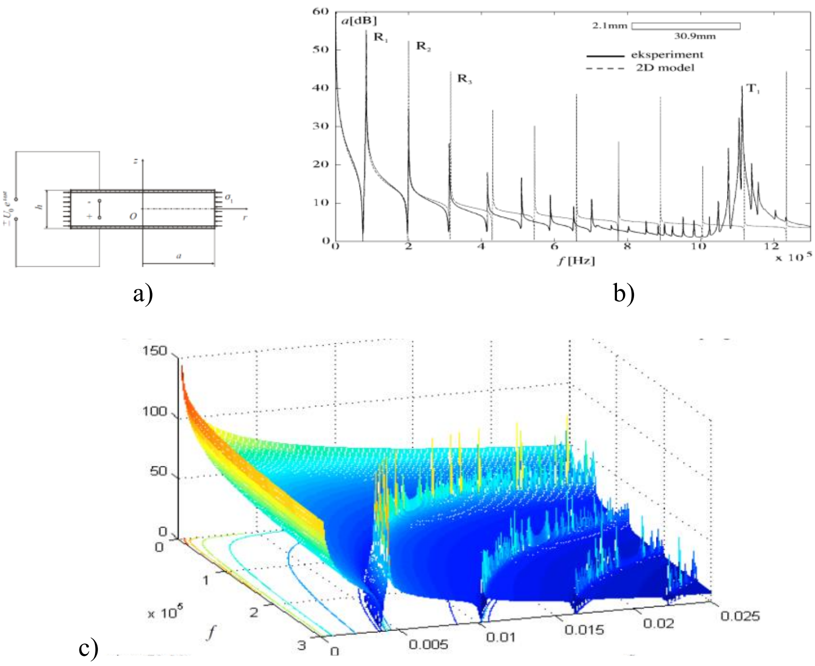


Fig. 27 a) Thin circular plate with transversal polarization and electrode coatings; b) Experimental and modelled character of impedance for PZT8 circular plate 2a=30,9[mm], h=2,1[mm]; c) Impedance of PZT8 piezoelectric circ. plate 2a=50[mm] with thickness h = 3[mm] in function of frequency and radius. Taken from [39]

Tables 1, 2, and 3, together with the corresponding crack deformation models shown in Figure 26, present the mechanical stress and strain states, as well as the piezoelectric states of the material in the vicinity of a semi-infinite crack for all three crack modes in a piezoelectric material. A biparametric analysis of the piezoelectric material state near the crack is also provided.

From the graphical comparison of different solution methods, it can be concluded that all approaches yield approximately equivalent results for a finite number of series terms, as demonstrated by the biparametric visualizations of the field state for the piezoceramic material (see Refs. [73, 74]). The method of applying complex variable functions [74] provides analytical solutions for n terms of the series as $n \to \infty$. Although these series are generally divergent, satisfactory accuracy can be achieved for a finite number of terms.

Chapter 8, "Problems of Oscillation in Electroelastic Piezoelectric Bodies," represents the central focus of this dissertation. It presents various analytical and numerical modeling methods for coupled electromechanical parameters, dynamic characteristics, and kinetic properties of piezoceramic materials, applied to piezoelectric bodies of different geometric shapes. Longitudinal oscillations of prismatic piezoceramic beams with longitudinal polarization and electrode coatings on the frontal surfaces are analyzed for two cases. This section also introduces a two-dimensional model of a thin piezoceramic circular plate (Fig. 27a) in an oscillatory state. The proposed model accurately predicts lower radial vibration modes of a thin circular plate and enables evaluation of the loading effect on the lateral circular surface. Consequently, it is possible to determine input electrical impedance, mechanical stresses, forces, displacements, specific strains, velocities, and accelerations at all points of the plate. A comparison between calculated and experimental results is provided. Frequency spectra are analyzed for different piezoceramic circular plates, along with the dependence of displacement along the plate radius. Similarly, the analysis of all characteristic parameters for thin piezoceramic circular plates, calculated in this doctoral dissertation [39], is presented (see Figs. 27 b) and c).

Table 1. Presentation of the results of the investigation of the mechanical and electrical state, in the vicinity of the tip of an infinite crack for mode I in a piezoelectric material with coupled mechanical and electric fields, taken from [39].

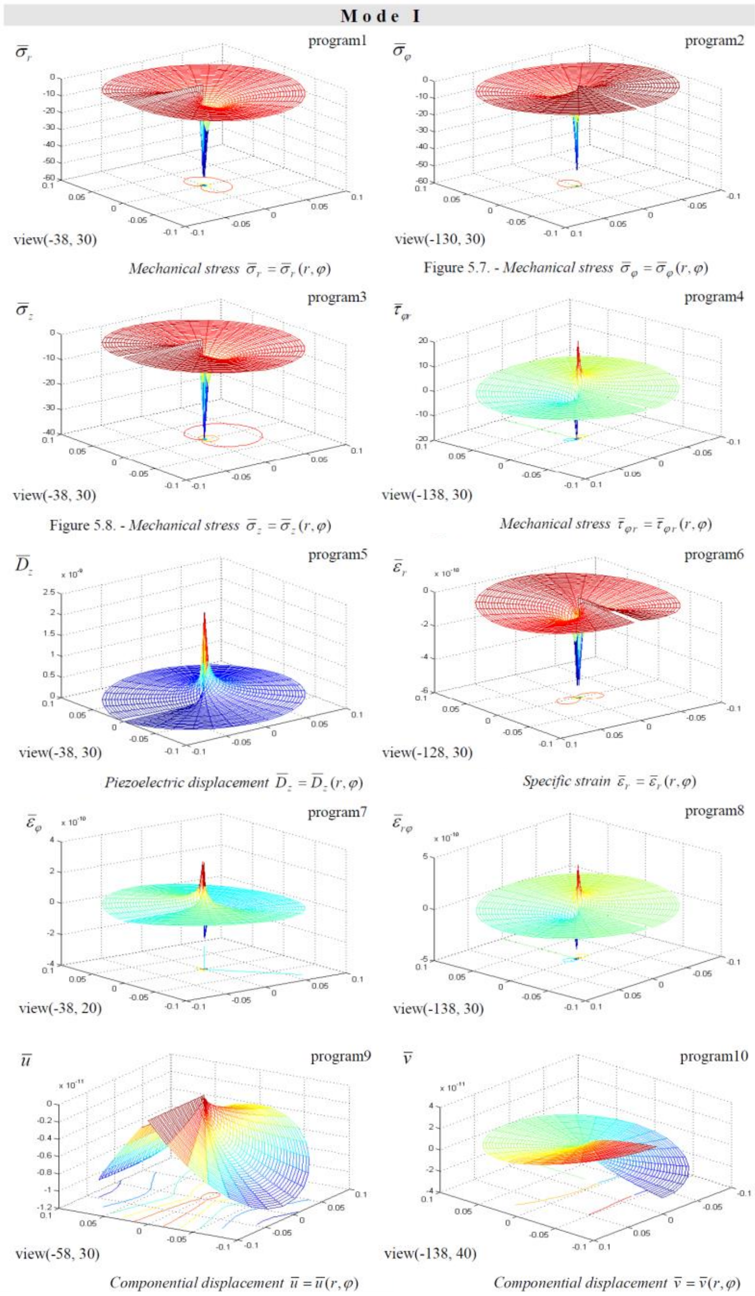


Table 2. Presentation of the results of the investigation of the mechanical and electrical state, in the vicinity of the tip of an infinite crack for mode II in a piezoelectric material with coupled mechanical and electric fields, taken from [39].

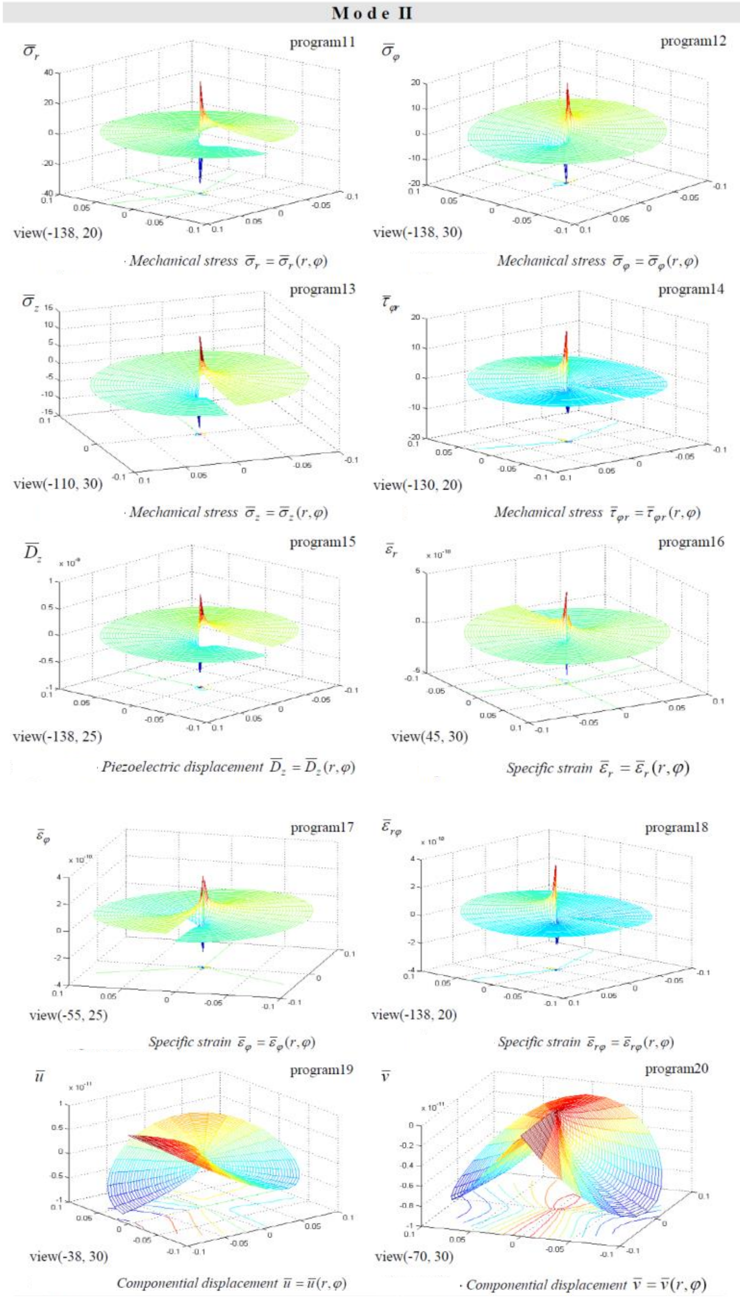
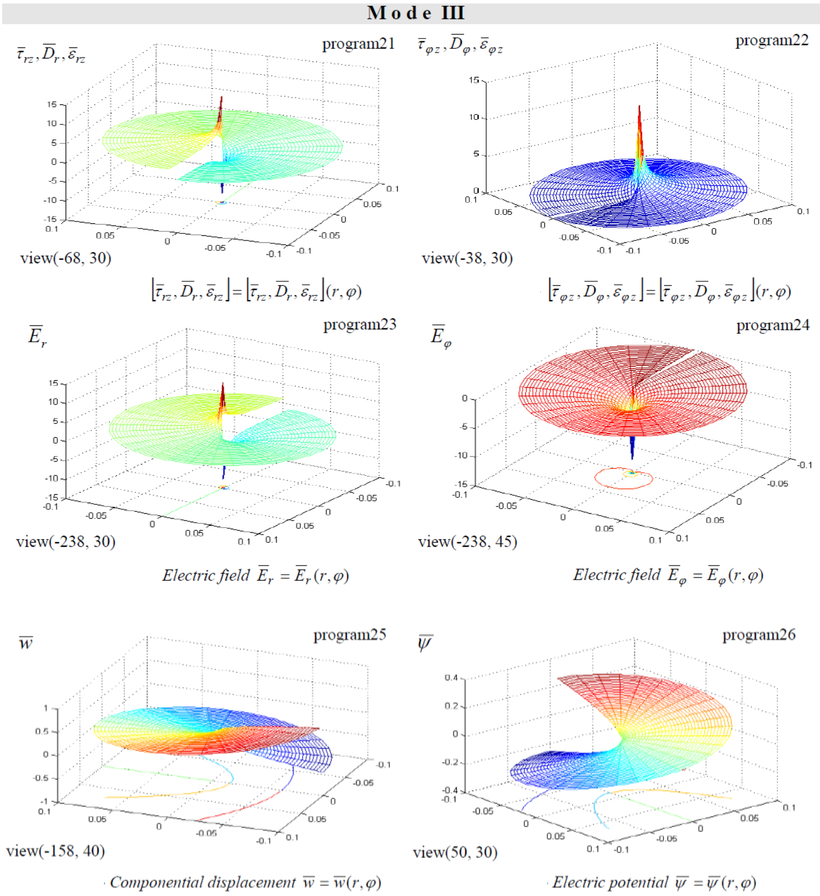


Table 3. Presentation of the results of the investigation of the mechanical and electrical state, in the vicinity of the tip of an infinite crack for mode III in a piezoelectric material with coupled mechanical and electric fields, taken from [39].



5. NEW CLASS OF COMPLEX RHEOLOGICAL MODELS OF MATERIALS WITH PIEZOELECTRIC PROPERTIES USING FRACTIONAL-ORDER DIFFERENTIAL CONSTITUTIVE RELATIONS AND A CLASS OF RHEOLOGICAL DISCRETE DYNAMIC SYSTEMS OF OSCILLATOR AND CREEPER TYPE

In this section, we present selected results of recent research concerning a new class of complex rheological models for materials with piezoelectric properties, characterized by fractional-order differential constitutive relations, as well as a new class of rheological discrete dynamic systems of oscillator and creeper types. This research is multidisciplinary, belonging to the domain of special continuum mechanics with elements of rheology, and finds applications in the dynamics of rheological discrete systems across various fields, including civil engineering, biological and biodynamic systems, and emerging technologies. The results related to discrete rheological models from [74] and fractional calculus from [76-78] served as the foundation for the latest findings published in papers [79–84].

Our aim is not to provide detailed derivations but to highlight key findings and refer readers to published papers [79–84], which contain comprehensive results and may serve as inspiration for further research. In this context, we present two comparative tables:

Table 4 summarizes models of a new class of rheological materials of fractional type with piezoelectric properties. For each model, fractional-order differential constitutive relations were established, along with characteristic properties such as stress relaxation and post-elasticity (see Refs. [79–84]).

Table 5 presents comparative models of rheological discrete dynamic systems, including fractional-type rheological oscillators and creepers (see Refs. [79–84]).

These tables provide insight into previously published scientific results and can inspire new research on the propagation of one-dimensional or spatial waves through rheological continua with fractional-order constitutive relations, or through piezoelectric rheological continua of fractional type. Research on continua with coupled fields has so far been scarcely addressed theoretically, and published results are practically innovative and new (see Refs. [79, 80]).

Further investigation of rheological discrete dynamic chains, both homogeneous and inhomogeneous of fractional type and with piezoelectric properties is of particular interest. For example, fractional-order differential constitutive relations for the Kelvin–Voigt–Faraday model of an elastopiezo-viscous solid and the Maxwell–Faraday model of a viscoelastic fluid are presented as follows:

$$\varepsilon_z + \frac{\boldsymbol{E}_\alpha}{\boldsymbol{E}+\boldsymbol{E}_e}\mathfrak{D}_t^\alpha[\varepsilon_z] = \frac{\sigma_z(t)}{\boldsymbol{E}+\boldsymbol{E}_e}\ ,\quad 0<\alpha\le 1$$

and

$$(\frac{1}{\boldsymbol{E}}+\frac{1}{\boldsymbol{E}_e})\mathfrak{D}_t^\alpha[\sigma_z(t)] + \frac{\sigma_z(t)}{\boldsymbol{E}_\alpha} = \mathfrak{D}_t^\alpha[\varepsilon_z(t)], 0<\alpha\le 1$$

Where

$$\mathfrak{D}_t^\alpha[x(t)] = \frac{d^\alpha x(t)}{dt^\alpha} = x^{(\alpha)}(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{x(\tau)}{(t-\tau)^\alpha}d\tau,\quad for\quad 0<\alpha\le 1$$

in which $\mathfrak{D}_t^\alpha[\bullet]$ the fractional order differential operator, α is the exponent of the fractional order differentiation in the interval $0<\alpha<1$. For details see Refs [79-81].

In the cited works [84], the following findings were demonstrated:

a) The rheological basic complex Kelvin–Voigt–Faraday (K-V-FY) model of a fractional-type elasto-viscous solid with piezoelectric properties and Faraday element polarization exhibits post-elastic behavior.

b) The rheological basic complex Maxwell–Faraday (M-FY) model of a fractional-type viscoelastic fluid with piezoelectric properties and Faraday element polarization exhibits normal stress relaxation.

Table 4. Tabular comparative overview of new class of complex rheological models of fractional types of ideal materials with piezoelectric properties

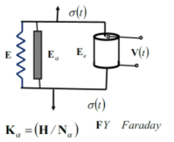
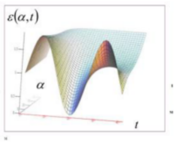
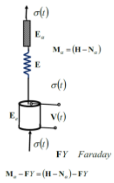
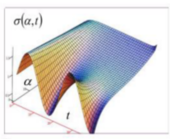
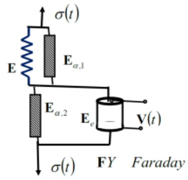
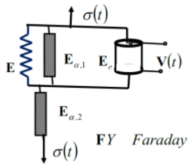
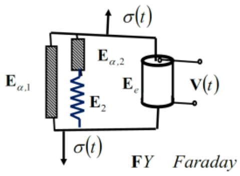
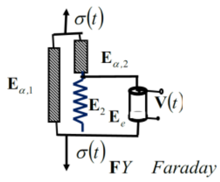
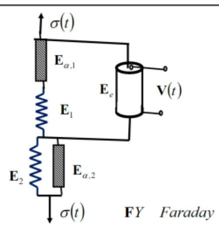
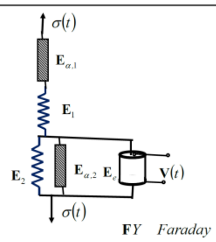
	Solid Elastoviscous fractional type piezoelectric property:	Fluid Visoelastic fractional type piezoelectric property:
Rheological basic complex model fractional type piezoelectric property: Kelvin-Voigt-Faraday's model and Maxwell-Faraday's model	$\mathbf{K}_\alpha = (\mathbf{H}/\mathbf{N}_\alpha)$ $\mathbf{K}_\alpha/\mathbf{F}Y = (\mathbf{H}/\mathbf{N}_\alpha)/\mathbf{F}Y$	$\mathbf{M}_\alpha - \mathbf{F}Y = (\mathbf{H} - \mathbf{N}_\alpha) - \mathbf{F}Y$
Two rheological complex model fractional type piezoelectric property: Lethersich – Faraday-F's moel and Lethersich–FaradayF's moel	$\mathbf{L}_\alpha = \mathbf{H}/\mathbf{N}_\alpha - \mathbf{N}_\alpha$ $\mathbf{L}_\alpha/\mathbf{F}Y = \mathbf{H}/\mathbf{N}_\alpha - \mathbf{N}_\alpha/\mathbf{F}Y$	$\mathbf{L}_\alpha = \mathbf{H}/\mathbf{N}_\alpha - \mathbf{N}_\alpha$ $\mathbf{L}_\alpha/\mathbf{F}Y = \mathbf{H}/\mathbf{N}_\alpha/\mathbf{F}Y - \mathbf{N}_\alpha$
Two rheological complex model fractional type piezoelectric property: Jeffrys-Faraday-F's model and Jeffrys -Faraday's model	$\mathbf{J}_\alpha/\mathbf{F}Y = \mathbf{N}_{\alpha,1}/(\mathbf{H} - \mathbf{N}_{\alpha,2})/\mathbf{F}Y$	$\mathbf{J}_\alpha/\mathbf{F}Y = \mathbf{N}_{\alpha,1}/(\mathbf{H}/\mathbf{F}Y - \mathbf{N}_{\alpha,2})$
Two rheological complex model fractional type piezoelectric property: Burgers – Faraday-F's model and Burgers-Faraday's model	$\mathbf{B}_{u,\alpha} - \mathbf{M}_\alpha/\mathbf{F}Y = \mathbf{K}_\alpha - \mathbf{M}_\alpha/\mathbf{F}Y$ $\mathbf{B}_{u,\alpha-} - \mathbf{M}_\alpha/\mathbf{F}Y = (\mathbf{H}/\mathbf{N}_\alpha) - (\mathbf{H} - \mathbf{N}_\alpha)/$	$\mathbf{B}_{Bu,\alpha}/\mathbf{F}Y = \mathbf{K}/\mathbf{F}Y - M_\alpha$ $\mathbf{B}_{Bu,\alpha}\mathbf{K}/\mathbf{F}Y = (\mathbf{H}/\mathbf{N}_\alpha/\mathbf{F}Y) - (\mathbf{H} - \mathbf{N}_\alpha$

Table 5. Tabular comparative overview of new class complex rheologic discrete dynamic systems of fractional type with one or finite number of degrees of freedom, [83].

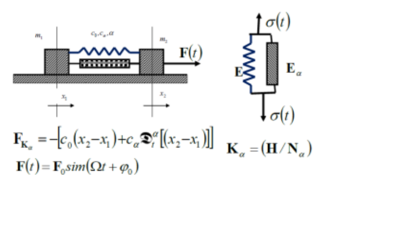
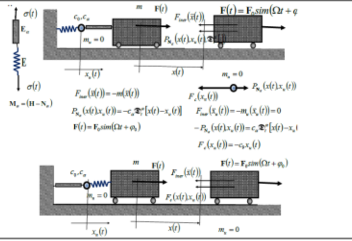
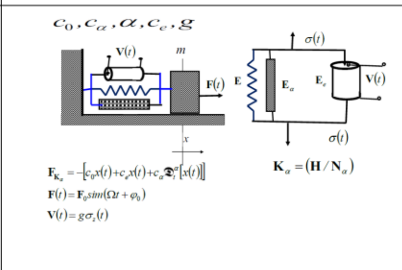
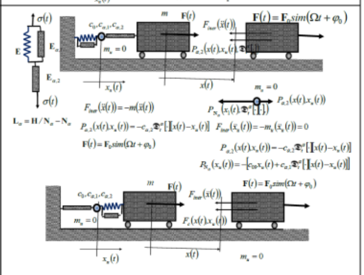
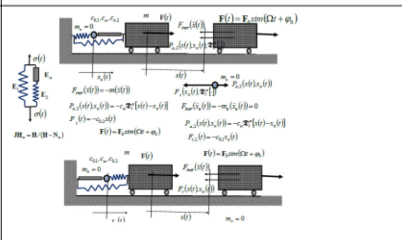
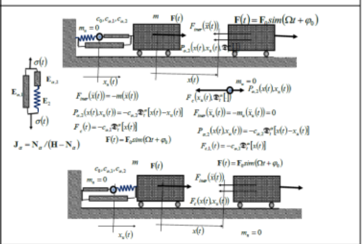
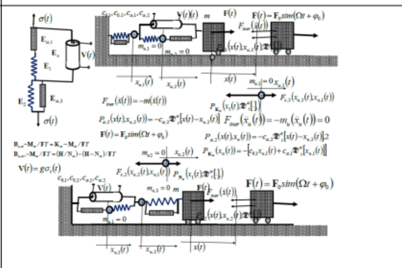
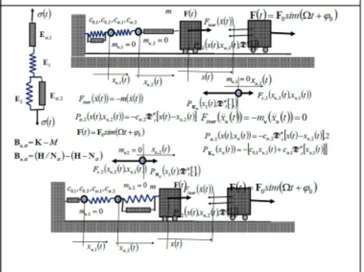
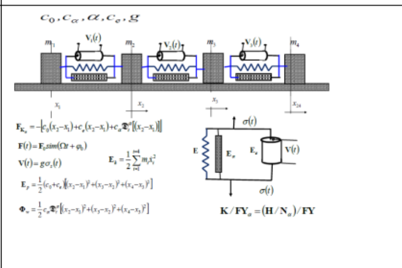
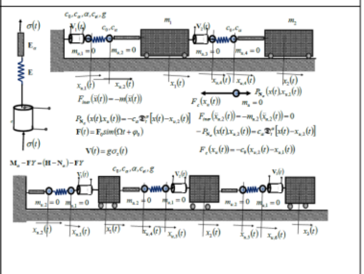
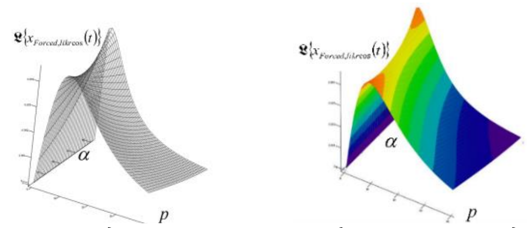
	Rheological oscillator fractional type	Rheological creperfractional type
A rheological Kelvin-Voigt discrete dynamic system of fractional-type rheological oscillator (left) and a rheological Maxwell discrete dynamic system, a fractional-type-rheological creeper(right)		
A rheological Kelvin-Voigt-Faraday discrete dynamic system of fractional-type with piezoelectric property-rheological oscillator (left) and a rheological Lethersich discrete dynamic system, a fractional-type-rheological creeper (right)		
A rheological Jeffrys-H discrete dynamic system of fractional-type rheological oscillator (left) and a rheological Jeffrys discrete dynamic system, a fractional-type-rheological creeper (right)		
A rheological Burgers-Faraday discrete dynamic system of fractional-type with piezoelectric property-rheological oscillator (left) and a rheological Burgers discrete dynamic system, a fractional-type-rheological creeper (right)		
A rheological Kelvin-Voigt-Faraday discrete dynamic system of fractional-type with piezoelectric property-rheological chain oscillator (left) and a rheological Maxwell-Faraday discrete dynamic system, a fractional-type-rheological chain creeper (right)		

Fig. 28 illustrates the surface plots of the Laplace transforms for the rheological elasto-viscous forced mode $L\{x_{\text{Forced, like cos}}\}$: (a) in black and white, and (b) in color. These plots correspond to the independent generalized coordinate $x(t)$ of the external degree of freedom in a basic rheological Kelvin-Voigt oscillator of fractional type. The system depends on the fractional differentiation order α (where $0 < \alpha < 1$) and the Laplace transform parameter p. The figure represents the elastoviscous oscillatory forced dynamics of the complex Kelvin-Voigt discrete system described in the upper part of Table 5, drawn using the corresponding analytical expression. The case shown is for a single-frequency periodic external force $F(t) = F_0\cos(\Omega t + \varphi_0)$, with external forced frequency $\Omega = 2$.



a- $\mathfrak{L}\{x_{Forced,likecos}\{\}\}$,$0 < \alpha < 1$ b- $\mathfrak{L}\{x_{Forced,likecos}\{\}\}$,$0 < \alpha < 1$

Fig. 28 Surface plots of Laplace transform for the elastoviscous forced mode $L\{x_{\text{Forced, like cos}}\}$: (a) black and white, (b) color. The plots correspond to the generalized coordinate $x(t)$of the external degree of freedom in a fractional Kelvin–Voigt oscillator ($0 < \alpha < 1$), depending on the fractional order $\alpha$and Laplace parameter p. The case shown is for a single-frequency periodic force $F(t) = F_0\cos(\Omega t + \varphi_0)$, with external forced frequency $\Omega = 2$.

6.CONCLUDING REMARKS

During the first half-century of scientific research and work at the Department of Mechanics, FMEN, highly successful Magister of Science programs were established under the leadership of the renowned Professor Dr. Eng. Dipl. Math D. P. Rašković. These programs featured seven advanced mathematics chapters-such as Tensor Calculus and Special Functions-along with subjects in Analytical Mechanics, Elastodynamics, and Nonlinear Oscillations. All examinations were conducted before a three-member commission of distinguished mechanics professors from universities across Serbia.

This curriculum provided a strong scientific foundation, equipping students with essential research methods for lifelong scientific work. A prerequisite for applying, researching, writing, and defending a doctoral dissertation was the previously earned title of Magister of Science. The first student to obtain this title in 1972-and later the Doctor of Science degree in 1975-was Katica Stevanović, exactly fifty years ago. The last graduates of this excellent system, who earned both Magister and Doctor of Science titles, were Dr. Dragan B. Jovanović in 2009 and Dr. Julijana D. Simonović in 2012, who later served as successive Heads of the Department of Mechanics.

Two strong schools of mechanics emerged from presented works and continue to thrive today:

The School of Continuum Mechanics (Theory of Elasticity, Fracture Mechanics, Mechanics of materials with coupled mechanical and electrical fields, and fractional-type materials) covered by research and results from projects 5P, 6P, 8P and 9P (see Appendix) and

The Serbian School of Nonlinear Oscillations and Dynamics of Hybrid Systems of Complex Structures covered by research and results from projects 1P-4P and 10P (see Appendix).

We firmly believe that both branches of mechanics at the Department of Mechanics, Faculty of Mechanical Engineering in Niš, will be carried forward by the research of young talents from new generations.

After half a century of this system, the Bologna Declaration introduced new doctoral programs in Serbian universities in 2010, from which dissertations continue today. However, these programs lack advanced mathematics chapters, focusing instead on numerical methods and commercial software packages-a strong global trend.

Building on the foundations of mechanics established by Professor Rašković and the two distinguished schools previously mentioned, the Department of Mechanics at the Faculty of Mechanical Engineering in Niš has sustained a tradition of excellence for more than five decades. This legacy has been strengthened by generations of talented individuals who have become leading professors and researchers throughout Serbia.Recent studies [85, 86] highlight the significance of precise analytical approaches in structural analysis. Rather than relying on general-purpose numerical methods such as FEM, the authors developed specialized analytical programs capable of calculating deflections at any point of a loaded beam. These tools deliver accurate solutions based on classical elasticity and structural mechanics, ensuring high reliability for engineering applications. Importantly, these studies are rooted in decades of expertise in mechanics of materials and fracture mechanics and were conducted in collaboration with Marija Stamenković Atanasov, a younger colleague who understands and continues the well-established tradition founded by Prof. Rašković. Such contributions reaffirm the enduring relevance of tailored analytical methods for practical design and optimization of machine elements and structures.

International collaboration has also been a constant throughout this journey. For example, working together Profs. K. Hedrih and D. B. Jovanović initiated cooperation with Fotios Georgiades (see [87]), which later evolved into numerous joint works (e.g., see [88]). Readers are encouraged to explore publication [89], which includes contributions from many internationally recognized scientists who participated in the conference *"Nonlinear Dynamics of Mechanical Systems"*, dedicated to the 75th birthday and 52 years of scientific contribution of Prof. Katica R. (Stevanović).

Over the past half-century, the Department of Mechanics has benefited from the contributions of numerous distinguished international scientists, including Academician Juri Alexejewitsch Mitropolski, Academician Tatomir Anđelić, Dragoslav Mitrinović, Dobrivoje Mihajlović, Vladimir Bogunović, Academician Božidar Vujanović, Jerzy T. Pindera, Academician Miloš Kojić, Emmanuel Gdoutos, Academician Felix Chernousko, Academician Antonios Kounadis, John Katsikadelis, Academician Richard Hetnarski, and many others.

Such a thorough and systematic approach to science, combined with strong international collaboration, opens boundaries and broadens scientific horizons, ensuring continuous progress and global impact.

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APPENDIX I – REFERENCES II -LIST OF PROJECTS

*Project Leader and SubProject Leader **Katica R. (Stevanović) Hedrih***
Mechanical Engineering Faculty University of Niš and Mathematica Institute SANU

1.P. Oscillations of the System with many degrees of Freedom and Elastic Bodies with Nonlinear Properties, Basic Scientific Found of Region Niš (1979-1981). Mechanical Engineering Faculty University of Niš .

2.P. Oscillations of the Special Elements and Systems, B Basic Scientific Found of Region Niš (1981-1986). Some research results included in two Magistar of sciences theses of P. Kozić and R. Pavlović. Mechanical Engineering Faculty University of Niš.

3.P. Stochastic Processes in Dynamical Systems-Applications on the Mechanical Engineering Systems, Basic Scientific Found of Region Niš (1986-1989). Some research results included in Magistar of sciences theses of Sl. Mitić and in two doctoral dissertations of P. Kozić and R. Pavlović. Mechanical Engineering Faculty University of Niš.

4.P. Nonlinear Deterministic and Stochastic Processes with Applications in Mechanical Engineering Systems, Ministry of Science and Technology Republic of Serbia, (1990-1995). Some research results included in two Magistar of sciences theses of Blagoj Pavlov and Aleksandar Filipovski and in a doctoral dissertation of Sl. Mitić. Mechanical Engineering Faculty University of Niš.

5.P. Sub-Projects 5.1. Thema: Stress and Strain State of the Deformable Bodies and 5.2. Theme: Vector Interpretation of Body Kinetic Parameters, as a part of Project: Actual Problems on Mechanic (1990-1995), Project Leader prof. dr Mane Šašić Ministry of Sciences, Technology and Development of Republic Serbia. Some research results included in three Magistar of sciences theses of Ljubiša Perić, Dragan Jovanović and Snežana Mitić. Mathematical Institute SANU.

6.P. Sub-Project: 04M03A Current Problems on Mechanics and Applications (1996-2000), as a part of Project: Methods and Models in Theoretical, Industrial and Applied Mathematics, Project Leader prof. dr Gradimir Milovanović, Ministry of Sciences, Technology and Development of Republic Serbia. Mathematica Institute SANU.

7.P. Project 1616 – Real Problems on Mechanics (2002-2004), Basic Science-Mathematics and Mechanics, ministry of Sciences, Technology and Development of

Republic Serbia. Some research results included in two doctoral dissertations of Ljubiša Perić and Dragan Jovanović. Mathematical Institute SANU and Mechanical Engineering Faculty University of Niš.

8.P. *Project ON1828 Dynamics and Control of active Structures (2001-2005),* Basic Science-Mathematics and Mechanics, Ministry of Sciences, Technology and Development of Republic Serbia. Some research results included in two doctoral dissertations of Ljubiša Perić and Dragan Jovanović. Mechanical Engineering Faculty University of Niš.

9.P. Project ON144002 -Theoretical and Applied Mechanics of the Rigid and Solid Bodies. Mechanics of Materials (2006-2010). Support: Ministry of Sciences and Environmental Protection of Republic of Serbia. Some research results included in two Magistar of sciences theses of Srdjan Jović, and Julijana Simonović and in four doctoral dissertations of Dragan Jovanović, Srdjan Jović, Ljiljana Veljović and Julijana Simonović. Institution Coordinator: Mathematical Institute Serbian Academy of Sciences and Arts and Mechanical Engineering Faculty University of Niš.

10.P. **ON174001 -** Dynamics of hybrid systems with complex structures. Mechanics of materials. (2011-2014), Ministry of Sciences and Technology of Republic of Serbia. Some research results included in three doctoral dissertations of Srdjan Jović, Ljiljana Veljović and Julijana Simonović. Institution Coordinator: Mathematical Institute Serbian Academy of Sciences and Arts and Mechanical Engineering Faculty University of Niš. The project team involved 42 researchers from mechanical engineering faculties from Niš, Beograd, Kragujevac, the Mathematics Institute of the Serbian Academy of Sciences and Arts, the Mathematics Faculty in Belgrade, the State University in Novi Pazar, the technological faculties in Leskovac and Bor, the Military -Technical Institute of Serbia, and other institutes in Belgrade. A number of 19 young researchers started doctoral studies and completed them by defending their PhD dissertations, among them the following: Danilo Karličić, Anđelka Hedrih, Ljubinko Kevac, Milan Cajić, Marija Mikić, Radivoje Radulović, Marija Stamenković Atanasov, Stepa Paunović, Nikola Nešić, Djordje Jovanović and others.