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TWO NEW BASIC RHEOLOGICAL DISCRETE DYNAMIC SYSTEMS OF THE FRACTIONAL TYPE OF PIEZOELECTRIC PROPERTY: THE KELVIN-VOIGT-FARADAY OSCILLATOR, AND THE MAXWELL-FARADAY CRAWLER

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Abstract. *We introduce two new fundamental rheological elements: a fractional-type Newtonian viscous element with a fractional-order differential constitutive relation and inherent energy dissipation, and a Faraday ideally elastic piezoelectric element characterized by electrical polarization induced by mechanical deformation. Using these elements, two new standard light rheological binder models are formulated: the standard light Kelvin-Voigt-Faraday fractional-type model and the standard light Maxwell-Faraday fractional-type model. Fractional-order differential constitutive relations are derived for both models. The former exhibits the property of delayed elasticity, while the latter demonstrates fractional stress relaxation.*

Based on these models, two discrete rheological dynamical systems with fractional-order behavior and piezoelectric properties are defined and their dynamics are investigated. According to their dynamic characteristics, the systems are termed the fractional Kelvin-Voigt-Faraday rheological oscillator and the fractional Maxwell-Faraday rheological crawler.

For the Kelvin-Voigt-Faraday oscillator, Laplace transforms and approximate analytical expressions for free and forced fractional-order oscillatory modes with piezoelectric coupling are obtained. For the Maxwell-Faraday crawler, Laplace transforms the independent generalized coordinates-one external and two internal-are derived. The analytical results are illustrated by appropriate graphical representations. Two theorems are formulated and proved.

In the concluding remarks, several analogies between fractional-type rheological mechanical systems and rheological electrical dynamical systems with piezoelectric properties are presented.

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1. INTRODUCTION

Piezoelectric materials are characterized by coupled tensors of mechanical and electrical states. In Refs. [3–6], the authors investigated the relationships between the tensor components of mechanical stress and deformation and the electrical polarization state for samples of various shapes and polarization directions, leading to important theoretical and experimental insights. In Refs. [7–10], the properties and operating principles of several devices employing piezoceramic tile exciters were presented. In such devices, a significant role is played by longitudinal oscillations in concentrators of rod-like form with variable cross-section, whose characteristics were studied and analyzed in Refs. [11–13].

Ultrasonic oscillations excited by piezoelectric transducers are widely used in the food industry, various metal-processing technologies, and ultrasonic cleaning applications. For example, a ten-minute experiment involving ultrasonic excitation of wine in a glass—performed with an ultrasonic exciter during the course of the Elastodynamics at the Faculty of Mechanical Engineering in Niš, over the period from 1978 to 2009—demonstrated an apparent aging effect of the wine, comparable to that observed in wine stored for a long time, on the order of a decade.

Today, the application of piezoelectric materials is widespread and continues to expand across numerous fields of technology and medicine, ranging from sensors and actuators to unconventional uses such as illuminated footwear soles, energy-harvesting floors in entertainment venues, and bridge-mounted systems for collecting electrical energy to supply lighting.

In this work, we introduce a new rheological element, referred to as the rheological Faraday ideally elastic element with polarization capability and piezoelectric properties, intended for the formation of complex rheological models of fractional-order materials. This element will be described in greater detail later (see Fig. 2d in the next chapter). Accordingly, in this introductory section, we provide an overview of existing publications on complex rheological models of fractional type, which are subsequently modified through serial or parallel coupling with the newly introduced rheological Faraday element.

A review of the literature indicates that rheological models are most frequently employed in the description of building materials, textile yarns, and various biomaterials. The majority of published studies on rheological modeling of ideal materials focus on classical complex rheological models, typically formulated in integral form, as summarized, for example, in a chapter of the university textbook on elasticity theory [14] (see Fig. 3 in the next chapter). Moreover, a substantial part of the literature addresses applications in construction engineering, particularly for concrete and rock materials, as well as in the textile industry, especially in processes related to cotton and yarn dyeing (see Refs. [15–20]). These conclusions are based on an extensive review of scientific publications and abstracts, analyzed using keywords such as *rheological material models*, *fractional-order constitutive relations*, and *fractional-type rheological dynamical systems*.

Classical linear rheological models, such as the Kelvin–Voigt and Maxwell models, as well as the modified Burgers model, have been widely employed in the literature. In Ref. [17], these models were applied to concrete in order to investigate its rheological behavior. A comprehensive overview of basic classical simple and complex rheological models is presented in Refs. [19, 20]. In particular, the classical Burgers model has been used to describe the elongation and dilation of yarns with different compositions, including wool, cotton, and cotton–wool blends.

More recent publications address biomaterials and the application of complex material models to biological media; however, these studies are still predominantly based on classical rheological frameworks, often supplemented by phenomenological modifications of a largely descriptive nature, rather than by rigorous mathematical developments. For the study of mechanical properties of biological materials, mathematical viscoelastic models of polymers, suspensions, and gels have been shown to be particularly useful [21].

The implementation of fractional derivatives in modeling viscoelastic, elastoviscous, and plastic material properties represents a modern and rapidly developing trend in science [22–24]. General fractional-order Kelvin–Voigt and Maxwell models are increasingly used to describe rheological phenomena in real materials exhibiting memory effects. Fractional flow models have been applied to characterize time-dependent behavior of non-Newtonian substances [24, 26], while a unified rheological model for cells and acellularized materials is presented in Ref. [27]. Data-driven approaches combining fractional rheology with neural networks have been proposed for identifying viscoelastic constitutive laws [28]. Within the fractional framework, thermomechanical models with memory effects have been compared with classical Volterra theory [29].

A complete and integral theory of analytical dynamics of discrete hereditary systems is presented in Ref. [30], including experimental procedures for identifying hereditary kernels of various rheological materials. Generalized fractional-type dissipation functions and extended matrix Lagrange equations are formulated in Ref. [31]. The authors have previously investigated the dynamics of discrete oscillatory systems with one or more degrees of freedom of fractional order, establishing analogies between mechanical and electrical fractional oscillators in linear and nonlinear regimes [12].

Although fractional-order differentiation dates back to the era of Leibniz and the Marquis de l'Hôpital, its widespread application in science and engineering has emerged only in recent decades. At present, several definitions of fractional differentiation exist, including those of Caputo, Riesz, Marchaud, Grünwald–Letnikov, and Riemann–Liouville. In this paper, we adopt Caputo's definition, which we introduce explicitly as the fractional-order differential operator employed throughout the analysis.

Within this framework, we introduce two new fundamental rheological elements: Newton's fractional-type viscous element, characterized by energy dissipation and a fractional-order constitutive relation, and Faraday's ideally elastic piezoelectric element, which exhibits electrical polarization induced by mechanical stress and deformation. Using these elements, we define two new standard lightweight complex rheological binding models for rheological discrete dynamical systems: the Kelvin–Voigt–Faraday model and the Maxwell–Faraday model, both of fractional type and possessing coupled mechanical and electrical (piezoelectric) properties. Fractional-order differential constitutive relations are formulated for each model.

By means of these standard lightweight rheological models, we construct two fundamental rheological discrete dynamical systems connecting a material point (rigid body) to a fixed point. The first system, termed the fractional Kelvin–Voigt–Faraday dynamical system with piezoelectric properties, represents an oscillator-type system with one degree of freedom. It exhibits fractional-order oscillatory motion with fractional-type energy dissipation, accompanied by electrical polarization and the corresponding electric voltage response.

For the defined rheological dynamical systems, one inhomogeneous fractional-order differential equation is derived for the oscillatory system, while three inhomogeneous fractional-order differential equations are formulated for the second system. A comprehensive dynamic analysis is carried out. The Laplace transforms of the independent generalized coordinates of both systems are obtained, and spatial surface representations are constructed in coordinate systems defined by the Laplace transform variable, the fractional-order differentiation exponent, and the Laplace transform parameter. This analysis is performed for both eigenmodes and forced modes under single-frequency periodic excitation.

For the rheological oscillator—the fractional Kelvin–Voigt–Faraday system with piezoelectric properties—approximate analytical time-domain expressions are derived for the eigenmodes and forced modes of the independent generalized coordinates. These include the translational displacement of the material point (rigid body), as well as the electric voltage and dielectric displacement of the Faraday element arising from mechanically induced electrical polarization. Several theorems formalizing these results are also established.

We use here the following definition of the derivation of the fractional order via the differential operator of the fractional order, which we denote by $\mathsf{D}_t^{\alpha}\left[\bullet\right]$ (for detail see Refs [22, 23, 24]).

2. TWO MODELS OF STANDARD LIGHT RHEOLOGIC BASIC COMPLEX IDEAL MODEL FRACTIONAL TYPE, PIEZOELECTRIC PROPERTY INTRODUCTION

New, rheologic ideally viscous fluid generalized Newton element of fractional type is with following differential constitutive relation fractional order in the following form:

$$\sigma_{z,\alpha} = \mathbf{E}_{\alpha}\mathsf{D}_t^{\alpha}\left[\varepsilon_z\right] \tag{1}$$

Such elements as basic rheologic Newton element of ideal viscous fluid element fractional type and Faraday's ideal elastic and piezoelectric element with pure ideal properties, can be combined into hybrid complex models, with Hooke-s ideal elastic element, where by one ser of elements of basic materials can be connected in two ways:
a*serial – in a series, which is indicated by a horizontal line "-" between the elements; and
b* parallel, which is indicated by a vertical line ‘/" between the elements.

Fig. 1 shows a series of classic rheological complex models, which are known from the literature, along next side two new series of modified rheological complex models, fraction type, and piezoelectric properties. In Fig. 3, five completes, each by three models and each containing: Classical complex rheological model of ideal materials with linear Newton’s element of viscous fluid, a new complex rheological model of ideal materials fractional type with new Newton’s element of viscous fluid, fractional type and a new complex rheological model of ideal materials fractional type with new Faraday's ideal elastic and piezoelectric element are presented (see Refs [1, 2, 3]). In details: a) Classical basic rheologic complex Kelvin-Voight model of elasto-viscous solid material; b)New basic rheologic complex Kelvin-Voight model, fractional type, of elasto-viscous solid material; c) rheologic Kelvin-Voight- Faraday's model, fractional type and piezoelectric property-solid body; d) Classical basic rheologic complex Maxwell model of elasto-viscous fluid material; e) New basic rheologic complex Maxwell model, fractional type, of elasto-viscous fluid material; f) New rheologic Maxwell-Faraday's model, fractional type and piezoelectric property, elasto-viscous fluid material; g) Classical basic rheologic complex Lethersich's model of elasto-viscous fluid material; h) New basic rheologic complex Lethersich's model, fractional type, of elasto-viscous fluid material; l) New rheologic Lethersich-Faraday's model, fractional type and piezoelectric property of elasto-viscous fluid material; n) Classical basic rheologic complex Jeffrey's model of elasto-viscous fluid material; m) New basic rheologic complex Jeffrey's model, fractional type, of elasto-viscous fluid material; p) New rheologic Jeffrey--Faraday's model, fractional type and piezoelectric property of elasto-viscous fluid material; r) Classical basic rheologic complex Burgers model of elasto-viscous fluid material; s) New basic rheologic complex Burgers model, fractional type, of elasto-viscous fluid material; t) New rheologic Burgers--Faraday's model, fractional type and piezoelectric property of elasto-viscous fluid material.

Fig. 2 shows the rheologic models of basic complex materials from three basic elements of ideal materials, fractional type and piezoelectric property. Modified rheologic basic complex Kelvin-Voigt- Faraday's model of the fractional type, piezoelectric property, presented in Figure 2.a, and denoted by $\mathbf{K}_{\alpha} / \mathbf{F}Y = (\mathbf{H} / \mathbf{N}_{\alpha}) / \mathbf{F}Y$. It is one of the two basic r complex rheologic models of ideal materials, created from three basic elements of ideal materials connected in parallel: Hooke's ideally elastic and modified rheologic Newton's viscous fluid fractional type and Faraday's element with piezoelectric property. see model presented at left part in Figure 2.a. Modified rheologic complex basic Maxwell-Faraday's model of fractional type, is one of the two basic complex models of materials, connected, three basic elements of ideal materials: Hooke's $\boldsymbol{H}$ ideally elastic and modified rheologic Newton's ideal fluid $\boldsymbol{N}_{\alpha}$ fractional type and Faraday's element with piezoelectric property, and has structural formula $\boldsymbol{M}_{\alpha} - \boldsymbol{F}Y = (\boldsymbol{H} - \boldsymbol{N}_{\alpha}) - \boldsymbol{F}Y$, see model presented at left part in Fig. 2.b.

In this work, we focus our research on two fundamental complex rheological models of fractional order with piezoelectric properties, considered as standard lightweight binding elements. These are:

(i) the standard lightweight complex rheological Kelvin–Voigt–Faraday model, a fractional-type viscoelastic solid with piezoelectric coupling, and
(ii) the standard lightweight complex rheological Maxwell–Faraday model, a fractional-type elastoviscous fluid with piezoelectric properties.

For both models, we present the corresponding fractional-order differential constitutive relations and analyze their fundamental mechanical and electromechanical properties.

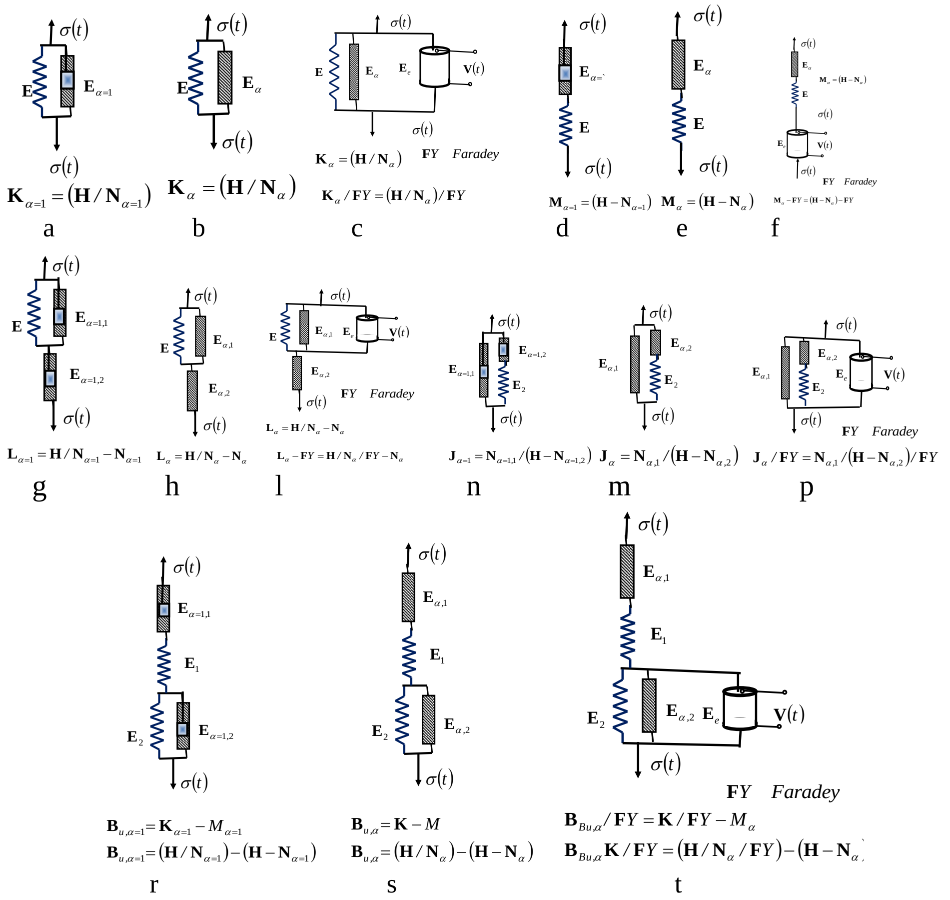
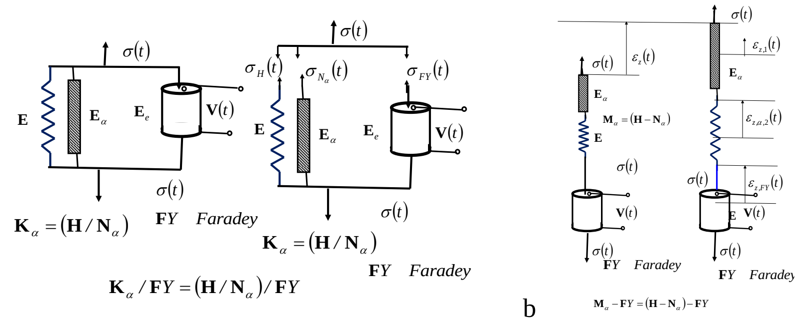


Fig. 1 Five completes each by three models and each containing: Classical complex rheological model of ideal materials with linear Newton's element of viscous fluid, a new complex rheological model of ideal materials fractional type with new Newton's element of viscous fluid, fractional type and a new complex rheological model of ideal materials fractional type with new Newton's element of viscous fluid, fractional type and a new complex rheological model of ideal materials fractional type with a new Faraday's ideal elastic and piezoelectric element

2.1. Differential constitutive relation of fractional order, new, standard light rheological basic complex Kelvin-Voigt-Faraday's model, fractional type with piezo-electric property

In the same Fig. 2.a, the structure of standard light rheologic basic complex Kelvin-Voigt-Faraday's, fractional type, model with piezoelectric property, of elastic-viscous solid material with decomposition is shown (right part of Fig. 4.a).

The decomposition of the model, from the right part of Line 4.a, gives the possibility of analyzing the component normal stresses and the resulting normal stress of that model.



a b

Fig. 2 New **g**eneralized complex basic rheological models, fractional type, of ideal materials with generalized Newton'a element, fractional type, of viscous fluid and Faraday's element piezoelectric property: a) Generalized rheologic basic complex Kelvin-Voigt- Faraday's, fractional type, model of elasto-viscous solid material with decomposition; b) Generalized rheologic complex Maxwell-Faraday's, fractional type, model of viscoelastic fluid piezoelectric property with decomposition of dilatation

The resulting normal stress $\sigma_z(t)$ at the ends of that standard light rheologic basic complex Kelvin-Voigt-Faraday's, fractional type, model with piezoelectric property, which is composed of a modified Kelvin-Voigt's fractional type model, in parallel connection with Faraday's ideal elastic piezoelectric property by the component elements, is equal to the sum of the component normal stresses of the rheological elements connected in parallel (see Fig. 2.a). The component normal stresses are of: Hooke's $\sigma_{z,H}(t)$, Newton's $\sigma_{z.\alpha}(t)$ and Faraday's $\sigma_{z,F}(t)$ elements, and their sum is equal to $\sigma_z(t)=\sigma_{z,H}(t)+\sigma_{z.\alpha}(t)+\sigma_{z,F}(t)$:

$$\sigma_z(t)=\mathbf{E}\varepsilon_z(t)+\mathbf{E}_\alpha \mathsf{D}_t^\alpha[\varepsilon_z(t)]+\mathbf{E}_e\varepsilon_z(t) \tag{2}$$

because the constitutive relations, the connection of normal stresses and axial dilatations, are individually for each of those rheological elements: $\sigma_{z,H}(t)=\mathbf{E}\varepsilon_z(t)$, $\sigma_{z.\alpha}(t)=\mathbf{E}_\alpha \mathsf{D}_t^\alpha[\varepsilon_z(t)]$ and $\sigma_{z,F}(t)=\mathbf{E}_e\varepsilon_z(t)$. In the preceding expressions, $\mathbf{E}$, $\mathbf{E}_e$ and $\mathbf{E}_\alpha$ are the material constants of the rheological elements in parallel connection: $\mathbf{E}$ Hooke's, $\mathbf{E}_e$ Newton's and $\mathbf{E}_\alpha$ Faraday's elements.

The electric voltage at the ends of Faraday's ideal elastic with piezoelectric properties in the complex model is:

$$\mathbf{V}_z(t)=-g\sigma_z(t) \tag{3}$$

Eq. (2) represents a differential constitutive relation of fractional order, and we can write it in the form:

$$\varepsilon_z+\frac{\mathbf{E}_\alpha}{\mathbf{E}+\mathbf{E}_e}\mathsf{D}_t^\alpha[\varepsilon_z]=\frac{\sigma_z(t)}{\mathbf{E}+\mathbf{E}_e} \tag{4}$$

This inhomogeneous differential equation of fractional order (4) represents the differential constitutive relation of the fractional order of the dependence of axial dilatation $\varepsilon_z(t)$ on normal stress $\sigma_z(t)$, model standard light rheologic basic complex Kelvin-Voigt-Faraday's, fractional type, model with piezoelectric property.

We can solve this inhomogeneous fractional differential Eq. (4) using the Laplace transform. By applying the Laplace transformation to the inhomogeneous differential equation of fractional order (4), we obtain the following algebraic equation:

$$\mathsf{L}\{\varepsilon_z\} = \frac{\mathsf{L}\{\sigma_z(t)\}}{(\mathbf{E}+\mathbf{E}_e)} \cdot \frac{1}{1+\dfrac{\mathbf{E}_\alpha}{\mathbf{E}+\mathbf{E}_e} p^\varepsilon} \tag{5}$$

It is an algebraic connection between the Laplace transform of axial dilatation $\mathsf{L}\{\varepsilon_z(t)\}$ and the Laplace transform of normal stress $\mathsf{L}\{\sigma_z(t)\}$, standard light rheologic basic complex Kelvin-Voigt-Faraday's, fractional type, model with piezoelectric property.

Now we develop the second factor from the previous Laplace transformation $\mathsf{L}\{\varepsilon_z(t)\}$ of the axial dilatation $\varepsilon_z(t)$, (5) in order by stems, using (see Reference [32]):

$$\frac{1}{1 \pm x} = 1 \mp x + x^2 \mp x^3 + x^4 \mp x^5 + \dots\dots\dots\dots \tag{6}$$

and since the expression $\dfrac{\mathbf{E}_\alpha}{\mathbf{E}+\mathbf{E}_e} p^\varepsilon$ is much smaller than one, i.e., as $\dfrac{\mathbf{E}_\alpha}{\mathbf{E}+\mathbf{E}_e} p^\varepsilon << 1$, we can expand the expression $\dfrac{1}{1+\dfrac{\mathbf{E}_\alpha}{\mathbf{E}+\mathbf{E}_e} p^\varepsilon}$ into degrees and write that:

$$\frac{1}{1+\dfrac{\mathbf{E}_\alpha}{\mathbf{E}+\mathbf{E}_e} p^\varepsilon} \approx \left\langle 1 + 1\sum_{k=1}^{\infty} (-1)^k \left(\frac{\mathbf{E}_\alpha}{(\mathbf{E}+\mathbf{E}_e)} \right)^k p^{k\alpha} \right\rangle \tag{7}$$

Now, the inverse Laplace transform of the previous expression (7) is obtained in the form of a function of time:

$$\mathsf{L}^{-1}\left\{ \frac{1}{1+\dfrac{\mathbf{E}_\alpha}{\mathbf{E}+\mathbf{E}_e} p^\varepsilon} \right\} \approx \mathsf{L}^{-1}\left\{ 1 + \sum_{k=1}^{\infty} (-1)^k \left(\frac{\mathbf{E}_\alpha}{(\mathbf{E}+\mathbf{E}_e)} \right)^k p^{k\alpha} \right\} = \sum_{k=0}^{\infty} (-1)^k \left(\frac{\mathbf{E}_\alpha}{\mathbf{E}+\mathbf{E}_e} \right)^k \frac{t^{(2-\alpha)k}}{\Gamma(2k+1-\alpha k)} \tag{8}$$

Based on the property of three functions, which are in convolution and for which it is valid that the product of the Laplace transforms of two functions is equal to the Laplace transform of the third in convolution, we can write:

$$\mathsf{L}\{\varepsilon_z(t)\} = \frac{\mathsf{L}\{\sigma_z(t)\}}{(\mathbf{E}+\mathbf{E}_e)} \cdot \frac{1}{1+\dfrac{\mathbf{E}_\alpha}{\mathbf{E}+\mathbf{E}_e} p^\varepsilon} = \frac{\mathsf{L}\{\sigma_z(t)\}}{(\mathbf{E}+\mathbf{E}_e)} \cdot \mathsf{L}\{f(t)\} \tag{9}$$

Based on the previous property of three functions in convolution (9) and the inverse Laplace transformation of the factor that we determined in the form (8), we can write the expression for axial dilation $\varepsilon_z(t)$ in the normal stress $\sigma_z(t)$ function using the integral of the convolution is in the form:

$$\varepsilon_z(t) \approx \frac{\sigma_z(t)}{(\mathbf{E}+\mathbf{E}_e)} * \left\langle \sum_{k=0}^{\infty} (-1)^k \left(\frac{\mathbf{E}_\alpha}{\mathbf{E}+\mathbf{E}_e} \right)^k \frac{t^{(2-\alpha)k}}{\Gamma(2k+1-\alpha k)} \right\rangle \tag{10}$$

Or in the form

$$\varepsilon_z(t) \approx \frac{1}{(\mathbf{E}+\mathbf{E}_e)} \int_0^t \sigma_z(t-\tau) \left\langle \sum_{k=0}^{\infty} (-1)^k \left(\frac{\mathbf{E}_\alpha}{\mathbf{E}+\mathbf{E}_e} \right)^k \frac{\tau^{(2-\alpha)k}}{\Gamma(2k+1-\alpha k)} \right\rangle d\tau \tag{11}$$

The two previous convolution integrals (10) and (11) represent the integral forms of the constitutive relationship of the connection of axial dilatation $\varepsilon_z(t)$ and normal stress $\sigma_z(t)$ of standard light rheologic basic complex Kelvin-Voigt- Faraday's, fractional type, model with piezoelectric property of polarization in Faraday's element.

Rheological Faraday's element, in of standard light rheologic basic complex Kelvin-Voigt- Faraday's, fractional type, model with piezoelectric property due to deformation and resulting axial dilatation $\varepsilon_z(t)$ and loading into the resulting normal stress $\sigma_z(t)$, passes into a state of polarization and is an analytical approximate expression for electric voltage, which appears in that element expressed in the form:

$$\mathbf{V}_z(t) = -g\sigma_z(t) \approx -g\int_0^t \sigma_z(t-\tau) \left\langle \sum_{k=0}^{\infty} (-1)^k \left(\frac{\mathbf{E}_\alpha}{\mathbf{E}+\mathbf{E}_e} \right)^k \frac{\tau^{(2-\alpha)k}}{\Gamma(2k+1-\alpha k)} \right\rangle d\tau \tag{12}$$

In the case of rest standard light rheologic basic complex Kelvin-Voigt-Faraday's, fractional type, model with piezoelectric property and at a very slow load change, when we can assume that the rate of expansion of the fractional type is small $\mathsf{D}_t^\alpha[\varepsilon_z] \to 0$ and tends to zero, the material behaves as a basic ideally elastic material, Hooke's and Faraday's and the normal stress of the material is almost proportional to the expansion $\sigma_{z\backslash} \to \mathbf{E}\varepsilon_z + \mathbf{E}_e\varepsilon_z$, respectively $\sigma_{z\backslash} \to (\mathbf{E}+\mathbf{E}_e)\varepsilon_z$.

$$\begin{gathered} \sigma_z = \sigma_{z,} + \sigma_{z.\alpha,2} + \sigma_{z,12} = \mathbf{E}\varepsilon_z + \mathbf{E}_\alpha \mathsf{D}_t^\alpha[\varepsilon_z] + \mathbf{E}_e\varepsilon_z \\ \mathsf{D}_t^\alpha[\varepsilon_z] \to 0 \quad \Rightarrow \quad \sigma_{z\backslash} \to \mathbf{E}\varepsilon_z + \mathbf{E}_e\varepsilon_z \quad \text{or} \quad \sigma_{z\backslash} \to (\mathbf{E}+\mathbf{E}_e)\varepsilon_z \\ \mathbf{V}_z(t) = -g\sigma_z(t) \end{gathered} \tag{13}$$

If the normal stress σ_z at the ends of a standard light rheologic basic complex Kelvin-Voigt- Faraday's, fractional type, model with piezoelectric property suddenly rises from zero to some finite value $\sigma_z = \sigma_{z,0} = const$, which remains constant in the following time interval, then we are interested in the behavior of the model of this basic model of the complex material.

If we assume that normal stress rises suddenly to some value and remains constant $\sigma_{z,0} = const$, then it is:

$$\mathbf{E}\varepsilon_z + \mathbf{E}_e\varepsilon_z + \mathbf{E}_\alpha \mathsf{D}_t^\alpha[\varepsilon_z] = \sigma_{z,0} = const \tag{14}$$

The preceding relation (14) is the differential constitutive relation of the fractional order of the deformation state standard light rheologic basic complex Kelvin-Voigt-Faraday's, fractional type, model with piezoelectric property under the action of constant normal stress $\sigma_{z,0} = const$ in the following time interval.

In order to find the time dependence of the dilatation $\varepsilon_z(t)$ of the standard light rheologic basic complex Kelvin-Voigt- Faraday's, fractional type, model with piezoelectric property, we first apply the Laplace transform to the previous constitutive relation-differential equation, fractional order (14), and obtain Laplace-ове трансформације аксијалне дилатације $\mathsf{L}\left\{\varepsilon_z(t)\right\}$:

$$\mathsf{L}\left\{\varepsilon_z\right\} = \frac{\sigma_{z,0}}{\left(\mathbf{E}+\mathbf{E}_e\right)} \cdot \frac{1}{p} \cdot \frac{1}{\left\langle 1 + \frac{\mathbf{E}_\alpha}{\left(\mathbf{E}+\mathbf{E}_e\right)} p^\alpha \right\rangle} \tag{15}$$

This Laplace transformation $\mathsf{L}\left\{\varepsilon_z(t)\right\}$ of axial dilation $\varepsilon_z(t)$ is the algebraic equation of the constitutive relationship of axial dilation $\varepsilon_z(t)$ and n constant stress $\sigma_{z,0} = const$ over the studied model.

Approximate analytical solution for axial dilatation $\varepsilon_z(t)$ as a function of time when the standard light rheologic basic complex Kelvin-Voigt-Faraday's, fractional type, model with piezoelectric property, is suddenly loaded with a constant normal stress $\sigma_{z,0} = const$ and under that stress in the following time interval is determined using inverse Laplace transformation $\varepsilon_z(t) = \mathsf{L}^{-1}\mathsf{L}\left\{\varepsilon_z\right\}\backslash$ from the last expression (15).

Now, it is necessary to determine the approximate analytical expression for the time function $\varepsilon_z(t) = \mathsf{L}^{-1}\mathsf{L}\left\{\varepsilon_z\right\}\backslash$, as the inverse Laplace transforms the previous expression and cut it into the time domain.

$$\varepsilon_z(t) = \mathsf{L}^{-1}\mathsf{L}\left\{\varepsilon_z\right\} = \mathsf{L}^{-1}\left\{ \frac{\sigma_{z,0}}{\left(\mathbf{E}+\mathbf{E}_e\right)} \cdot \frac{1}{p} \cdot \frac{1}{\left\langle 1 + \frac{\mathbf{E}_\alpha}{\left(\mathbf{E}+\mathbf{E}_e\right)} p^\alpha \right\rangle} \right\} \tag{16}$$

That is why it is necessary to develop the expression $\frac{\sigma_{z,0}}{\left(\mathbf{E}+\mathbf{E}_e\right)} \cdot \frac{1}{p} \cdot \frac{1}{\left\langle 1 + \frac{\mathbf{E}_\alpha}{\left(\mathbf{E}+\mathbf{E}_e\right)} p^\alpha \right\rangle}$ in order by powers of p, which is a complex number, using the form (6), and we obtain the following expression:

$$\frac{\sigma_{z,0}}{\left(\mathbf{E}+\mathbf{E}_e\right)} \cdot \frac{1}{p} \cdot \frac{1}{\left\langle 1 + \frac{\mathbf{E}_\alpha}{\left(\mathbf{E}+\mathbf{E}_e\right)} p^\alpha \right\rangle} = \frac{\sigma_{z,0}}{\left(\mathbf{E}+\mathbf{E}_e\right)} \cdot \frac{1}{p} \cdot \left\langle 1 + \sum_{k=1}^{\propto} (-1)^k \left(\frac{\mathbf{E}_\alpha}{\left(\mathbf{E}+\mathbf{E}_e\right)} p^\alpha \right)^k \right\rangle \tag{17}$$

The inverse Laplace transform $\varepsilon_z(t) = \mathsf{L}^{-1}\mathsf{L}\left\{\varepsilon_z\right\}\backslash$ now gives an analytically approximate expression for the time-domain axial dilation $\varepsilon_z(t)$ for the standard light rheologic basic complex Kelvin-Voigt- Faraday's, fractional type, model with piezoelectric property, model, when suddenly subjected to a constant normal stress $\sigma_{z,0}$ and kept under constant normal stress in the following form:

$$\varepsilon_z(t)=\mathsf{L}^{-1}\mathsf{L}\{\varepsilon_z\}=\frac{\sigma_{z,0}}{(\mathbf{E}+\mathbf{E}_e)}\cdot\cdot\mathsf{L}^{-1}\left\{\frac{1}{p}\cdot\left\langle 1+\sum_{k=1}^{\propto}(-1)^k\left(\frac{\mathbf{E}_\alpha}{(\mathbf{E}+\mathbf{E}_e)}\right)^k p^{k\alpha-1}\right\rangle\right\} \quad (18)$$

$$\mathbf{V}_z(t)=-g\sigma_z(t)$$

The inverse Laplace transform (18) now gives an analytically approximate expression for the dilatation $\varepsilon_z(t)$ in a Faraday element in the standard light rheologic basic complex Kelvin-Voigt-Faraday's, fractional type, model with piezoelectric property, in that case suddenly subjected to a constant normal stress $\sigma_{z,0}=const$ in the time domain:

$$\varepsilon_z(t)=\mathsf{L}^{-1}\mathsf{L}\{\varepsilon_z\}=\frac{\sigma_{z,0}}{(\mathbf{E}+\mathbf{E}_e)}\cdot\left\langle 1+\sum_{k=1}^{\propto}(-1)^k\left(\frac{\mathbf{E}_\alpha}{(\mathbf{E}+\mathbf{E}_e)}\right)^k\frac{t^{(2-\alpha)k+1}}{\Gamma(2k+2-\alpha k)}\right\rangle \quad (19)$$

$$\varepsilon_z(t)=\frac{\sigma_{z,0}}{(\mathbf{E}+\mathbf{E}_e)}\cdot\left\langle 1+\sum_{k=1}^{\propto}(-1)^k\left(\frac{\mathbf{E}_\alpha}{(\mathbf{E}+\mathbf{E}_e)}\right)^k\frac{t^{(2-\alpha)k+1}}{\Gamma(2k+2-\alpha k)}\right\rangle$$

The electric polarization voltage in the Faraday element in the standard light rheologic basic complex Kelvin-Voigt- Faraday's, fractional type, model with piezoelectric property, in that case suddenly subjected to a constant normal stress $\sigma_{z,0}=const$, is:

$$\mathbf{V}_z(t)=-g\sigma_{z,0}\cdot\left\langle 1+\sum_{k=1}^{\propto}(-1)^k\left(\frac{\mathbf{E}_\alpha}{(\mathbf{E}+\mathbf{E}_e)}\right)^k\frac{t^{(2-\alpha)k+1}}{\Gamma(2k+2-\alpha k)}\right\rangle \quad (20)$$

The dielectric displacement, D, $D_z(t)=b\sigma_z(t)$ and $D_z(t)=e\varepsilon_z(t)$, in the Faraday element in the standard light rheologic basic complex Kelvin-Voigt- Faraday's, fractional type, model with piezoelectric property, in that case suddenly subjected to a constant normal stress $\sigma_{z,0}=const$, is:

$$D_z(t)=e\varepsilon_z(t)=e\frac{\sigma_{z,0}}{(\mathbf{E}+\mathbf{E}_e)}\cdot\cdot\left\langle 1+\sum_{k=1}^{\propto}(-1)^k\left(\frac{\mathbf{E}_\alpha}{(\mathbf{E}+\mathbf{E}_e)}\right)^k\frac{t^{(2-\alpha)k+1}}{\Gamma(2k+2-\alpha k)}\right\rangle$$

(21)

This model, the standard light rheologic basic complex Kelvin-Voigt-Faraday's, fractional type, model with piezoelectric property, with the structure of parallel connection of the basic rheological elements, has the characteristic of subsequent elasticity, when axial dilation lags behind normal stress.

2.2. Differential constitutive relation of fractional order, new, standard light rheological basic complex Maxwell-Faraday's model, fractional type with piezo-electric property

In the same Fig. 2.b, the structure of the standard light rheologic basic complex Maxwell-Faraday's, fractional type, model with piezoelectric property, of elasto-viscous fluid material with decomposition by component axial dilatations (right part of Fig. 2.b) is shown.

The decomposition of the components of the axial dilatations of the rheological elements of the model, from the right part of Fig. 2.b, gives the possibility of analyzing the component axial dilatations of the Hooke elemet $\varepsilon_{z,H}=\varepsilon_{z,H,1}$, the fractional type Newton element $\varepsilon_{z,\alpha}=\varepsilon_{z,\alpha,2}$ and the Faraday element $\varepsilon_{z,F}=\varepsilon_{z,F,3}$ and the resulting axial dilatation $\varepsilon_z(t)$ of that model.

Modified rheological basic standard light Maxwell's model, fractional type $\boldsymbol{M}_\alpha$, whose structural formula is $\boldsymbol{M}_\alpha = (\boldsymbol{H} - \boldsymbol{N}_\alpha)$, when entered into its structure, in a regular connection, Faraday's ideal elastic and piezoelectric element gives the standard light rheologic basic complex Maxwell-Faraday's, fractional type, model with piezoelectric property.

That regular-serially connection of Hooke's ideally elastic element, and Newton's viscous element, fractional type, and Faraday's ideal elastic and piezoelectric element, has the property that, throughout the standard light rheologic basic complex Maxwell-Faraday's, fractional type, model with piezoelectric property $\boldsymbol{M}_\alpha - \boldsymbol{F}Y = (\boldsymbol{H} - \boldsymbol{N}_\alpha) - \boldsymbol{F}Y$, the velocity of the resulting axial dilatation of the fractional type $D_t^\alpha[\varepsilon_z]$, equal to the sum of the component velocities, fractional type, axial dilatations of the elements in a series connection, $D_t^\alpha[\varepsilon_{z.H}] = \dfrac{1}{\mathbf{E}} D_t^\alpha[\sigma_z]$ Hooke's element, $D_t^\alpha[\varepsilon_{z.\alpha}] = \dfrac{\sigma_z}{\mathbf{E}_\alpha}$ Newton's element of the fractional type and $D_t^\alpha[\varepsilon_{z.F}] = \dfrac{1}{\mathbf{E}_e} D_t^\alpha[\sigma_z]$ Faraday's element of ideal elastic and piezoelectric properties.

The normal stress is the same throughout the entire standard light rheological complex Maxwell-Faraday model, of the fractional type, with the property of piezoelectricity, so we can write the constitutive relations of each basic rheological element in this series connection, so the нормал stresses are a function of dilation: of Hooke's element $\sigma_{z,H} = \mathbf{E}\varepsilon_{z,H}$, of Newton's element fractional type $\sigma_{z,\alpha} = \mathbf{E}_\alpha D_t^\alpha[\varepsilon_{z,\alpha}]$ and

$\sigma_{z,F} = \mathbf{E}_e \varepsilon_{z,F}$ Faraday's element ideal elastic with piezoelectric property.

The rate (velocity) of dilation $D_t^\alpha[\varepsilon_z]$ of the standard light rheological complex Maxwell-Faraday model, of the fractional type, with the property of piezoelectricity is equal to the sum of the rates of dilation of the fractional type $D_t^\alpha[\varepsilon_{z.\alpha}] = \dfrac{\sigma_z}{\mathbf{E}_\alpha}$ Newton's viscous element, fractional type, $D_t^\alpha[\varepsilon_{z.H}] = \dfrac{1}{\mathbf{E}} D_t^\alpha[\sigma_z]$ Hooke's ideally elastic element and $D_t^\alpha[\varepsilon_{z.F}] = \dfrac{1}{\mathbf{E}_e} D_t^\alpha[\sigma_z]$ Faraday's element, with the property of piezoelectricity, in the form of:

$$D_t^\alpha[\varepsilon_z] = D_t^\alpha[\varepsilon_{z,H}] + D_t^\alpha[\varepsilon_{z,\alpha}] + D_t^\alpha[\varepsilon_{z,F}] \tag{23.a}$$

it follows that this sum is in the following form (see right part in Figure 2.б):

$$\mathsf{D}_t^{\alpha}\left[\varepsilon_z\right]=\frac{1}{\mathbf{E}}\mathsf{D}_t^{\alpha}\left[\sigma_z\right]+\frac{\sigma_z}{\mathbf{E}_{\alpha}}+\frac{1}{\mathbf{E}_e}\mathsf{D}_t^{\alpha}\left[\sigma_z\right] \tag{23.b}$$

We write the previous differential constitutive relation of fractional order (23) in the following standard form:

$$\mathsf{D}_t^{\alpha}\left[\sigma_z(t)\right]+\frac{\sigma_z(t)}{\mathbf{E}_{\alpha}\left(\dfrac{1}{\mathbf{E}}+\dfrac{1}{\mathbf{E}_e}\right)}=\frac{1}{\left(\dfrac{1}{\mathbf{E}}+\dfrac{1}{\mathbf{E}_e}\right)}\mathsf{D}_t^{\alpha}\left[\varepsilon_z(t)\right]\neq const \tag{24}$$

The previous inhomogeneous differential equation of fractional order (24) gives a differential relation of fractional order between axial dilatation $\varepsilon_z(t)$ and normal stress $\sigma_z(t)$ of standard light rheological complex Maxwell-Faraday model, of the fractional type, with the property of piezoelectricity.

Then, we apply the Laplace transformation to that inhomogeneous differential constitutive relation, of the fractional type (24), and after applying the Laplace transformation, we write the previous relation in the following form:

$$\mathsf{L}\left\{\sigma_z(t)\right\}=\frac{\mathbf{E}_{\alpha}p^{\alpha}}{\left\langle 1+\mathbf{E}_{\alpha}\left(\dfrac{1}{\mathbf{E}}+\dfrac{1}{\mathbf{E}_e}\right)p^{\alpha}\right\rangle}\mathsf{L}\left\{\varepsilon_z(t)\right\}\neq const \tag{25}$$

It is an algebraic connection (25) between Laplace transformations $\mathsf{L}\left\{\sigma_z\right\}$ of normal stress $\sigma_z(t)$ and $\mathsf{L}\left\{\varepsilon_z(t)\right\}$ axial dilatation $\varepsilon_z(t)$ of standard light rheological complex Maxwell-Faraday model, of the fractional type, with the property of piezoelectricity.

We can write the previous Alebarian equation (25) in the following form:

$$\mathsf{L}\left\{\sigma_z(t)\right\}=\mathbf{E}_{\alpha}\mathsf{L}\left\{\varepsilon_z(t)\right\}\cdot\left[p^{\alpha}\left\langle 1+\sum_{k=1}^{\propto}(-1)^k\left(\frac{\mathbf{E}_{\alpha}}{\mathbf{E}_e}+\frac{\mathbf{E}_{\alpha}}{\mathbf{E}}\right)^k p^{k\alpha}\right\rangle\right] \tag{26}$$

in which we expanded the second factor from (25) into a power-order p, using the development of (6) for small values of $\left(\dfrac{\mathbf{E}_{\alpha}}{\mathbf{E}_e}+\dfrac{\mathbf{E}_{\alpha}}{\mathbf{E}}\right)p^{\alpha}<<1$, in the form:

$$\frac{p^{\alpha}}{\left\langle 1+\left(\dfrac{\mathbf{E}_{\alpha}}{\mathbf{E}_e}+\dfrac{\mathbf{E}_{\alpha}}{\mathbf{E}}\right)p^{\alpha}\right\rangle}=p^{\alpha}\frac{1}{\left\langle 1+\left(\dfrac{\mathbf{E}_{\alpha}}{\mathbf{E}_e}+\dfrac{\mathbf{E}_{\alpha}}{\mathbf{E}}\right)p^{\alpha}\right\rangle}\approx p^{\alpha}\cdot\left\langle 1+\sum_{k=1}^{\propto}(-1)^k\left(\frac{\mathbf{E}_{\alpha}}{\mathbf{E}_e}+\frac{\mathbf{E}_{\alpha}}{\mathbf{E}}\right)^k p^{k\alpha}\right\rangle \tag{27}$$

The analytical approximate solution of the algebraic Eq. (25) is obtained by the inverse Laplace transformation of the expression (26) and the convolution integral in the form:

$$\sigma_z(t)=\mathbf{E}_{\alpha}\mathsf{L}^{-1}\mathsf{L}\left\{\varepsilon_z(t)\right\}*\mathsf{L}^{-1}\left[p^{\alpha}\left\langle 1+\sum_{k=1}^{\propto}(-1)^k\left(\frac{\mathbf{E}_{\alpha}}{\mathbf{E}_e}+\frac{\mathbf{E}_{\alpha}}{\mathbf{E}}\right)^k p^{k\alpha}\right\rangle\right] \tag{28.a}$$

$$\sigma_z(t) = \mathbf{E}_\alpha \varepsilon_z(t) * \mathsf{L}^{-1}\left[p^\alpha \left\langle 1 + \sum_{k=1}^{\propto} (-1)^k \left(\frac{\mathbf{E}_\alpha}{\mathbf{E}_e} + \frac{\mathbf{E}_\alpha}{\mathbf{E}} \right)^k p^{k\alpha} \right\rangle \right] \qquad (28.b)$$

or

$$\sigma_z(t) = \mathbf{E}_\alpha \int_0^t \varepsilon_z(t-\tau) \left\langle \frac{t^\alpha}{\Gamma(2-\alpha)} + \sum_{k=1}^{\propto} (-1)^k \left(\frac{\mathbf{E}_\alpha}{(\mathbf{E}+\mathbf{E}_e)} \right)^k \frac{\tau^{(3-\alpha)k+1}}{\Gamma(3k+2-\alpha k)} \right\rangle d\tau \qquad (29)$$

Let's now study the property of standard light rheological complex Maxwell-Faraday model, of the fractional type, with the property of piezoelectricity, when the rate of normal stress, fractional type $\mathsf{D}_t^\alpha[\sigma_z]$ change tends to zero, $\mathsf{D}_t^\alpha[\sigma_z] \to 0$ the model behaves like a viscoelastic fluid, because the deformation, that is, the axial dilatation ε_z of the fluid type, grows indefinitely without increasing the load. When the model of standard light rheological complex Maxwell-Faraday model, of the fractional type, with the property of piezoelectricity, is unloaded, the deformation in Hooke's ideally elastic element and Faraday's piezoelectric element completely disappears, while the deformation due to flow in with a modified Newton's viscous element-viscous fluid, of the fractional type in a regular connection, does not disappear.

When the rate of normal stress $\mathsf{D}_t^\alpha[\sigma_z]$ of the fractional type of change in the normal stress tends to zero, $\mathsf{D}_t^\alpha[\sigma_z] \to 0$, the model of the standard light rheological complex Maxwell-Faraday model, of the fractional type, with the property of piezoelectricity behaves like a viscoelastic fluid because: $\sigma_z \to \mathbf{E}_\alpha \mathsf{D}_t^\alpha[\varepsilon_z]$ workload. When the model of the standard light rheological complex Maxwell-Faraday model, of the fractional type, with the property of piezoelectricity, is unloaded, the deformation in Hooke's ideally elastic element and Faradey's ideally elastic and piezo-electric element completely disappears, while the deformation due to flow in the modified Newton's viscous element-viscous fluid, fractional type in regular-serial connection, does not disappear.

If this model of the standard light rheological complex Maxwell-Faraday model, of the fractional type, with the property of piezoelectricity,, is suddenly loaded to a certain value of the normal stress $\sigma_{z,0}$, it will respond to an elastic deformation $\varepsilon_{z,0} = \dfrac{\sigma_{z,0}}{\mathbf{E}+\mathbf{E}_e}$, created instantaneously in Hooke's ideally elastic element and Faraday's piezo- electric element, because due to a sudden load, immediately at the beginning of the observation of the behavior of the model in the standard light rheological complex Maxwell-Faraday model, of the fractional type, with the property of piezoelectricity, flow in the regularly-serially connected modified Newton's viscous element, fractional type - ideally viscous fluid, fractional type does not come to the fore. If we prevent the development of deformation-dilation, assuming that the rate of dilation, of the fractional type, tends to zero, $\mathsf{D}_t^\alpha[\varepsilon_z] \to 0$, then the normal stress is a function of time, which needs to be determined.

When the normal stress rate $\mathsf{D}_t^{\alpha}\left[\sigma_z\right]$ of the fractional type of the standard light rheological complex Maxwell-Faraday model, of the fractional type, with the property of piezoelectricity, tends to zero, $\mathsf{D}_t^{\alpha}\left[\sigma_z\right]\to 0$, then the normal mechanical stress tends to a value proportional to the dilation rate (velocity) of the fractional type: $\sigma_z \to \mathbf{E}_{\alpha}\mathsf{D}_t^{\alpha}\left[\varepsilon_z\right]$:

$$\mathsf{D}_t^{\alpha}\left[\sigma_z\right]\to 0 \Rightarrow \qquad \sigma_z \to \mathbf{E}_{\alpha}\mathsf{D}_t^{\alpha}\left[\varepsilon_z\right] \tag{30}$$

In order to determine the functional dependence of the normal stress on time, when we keep the model of the standard light rheological complex Maxwell-Faraday model, of the fractional type, with the property of piezoelectricity, at some constant rate (velocity) of axial dilatation, fractional type, $\left\{\mathsf{D}_t^{\alpha}\left[\varepsilon_z\right]_{z,,0}\right\}=const$, and by use differential constitutive relation fractional order (24),we write that:

$$\frac{1}{\mathbf{E}}\mathsf{D}_t^{\alpha}\left[\sigma_z\right]+\frac{1}{\mathbf{E}_e}\mathsf{D}_t^{\alpha}\left[\sigma_z\right]+\frac{\sigma_z}{\mathbf{E}_{\alpha}}=\left\{\mathsf{D}_t^{\alpha}\left[\varepsilon_z\right]_{z,,0}\right\}=const \tag{31}$$

Then, we apply the Laplace transformation to that functional dependence (31), fractional type, which is differential constitutive relation, fractional order of consider dynamic state of the standard light rheological complex Maxwell-Faraday model, of the fractional type, with the property of piezoelectricity, and after applying the Laplace transformation to the previous relation (31), we get:

$$\mathsf{L}\left\{\sigma_z\right\}=\mathbf{E}_{\alpha}\left\{\mathsf{D}_t^{\alpha}\left[\varepsilon_z\right]_{z,,0}\right\}\frac{1}{p}\cdot\frac{1}{\left\langle 1+\left(\frac{\mathbf{E}_{\alpha}}{\mathbf{E}_e}+\frac{\mathbf{E}_{\alpha}}{\mathbf{E}}\right)p^{\alpha}\right\rangle} \tag{32}$$

The previous relation (32) is the algebraic connection of the Laplace transformation $\mathsf{L}\left\{\sigma_z(t)\right\}$ of the normal stress $\sigma_z(t)$ in the model of the standard light rheological complex Maxwell-Faraday model, of the fractional type, with the property of piezoelectricity in conditions of prevented further development of axial dilatation $\left\{\mathsf{D}_t^{\alpha}\left[\varepsilon_z\right]_{z,,0}\right\}=const$.

Now, it is necessary to determine an approximate analytical expression for the normal stress $\sigma_z(t)$ as a function of time, in a the standard light rheological complex Maxwell-Faraday model, of the fractional type, with the property of piezoelectricity, as the inverse Laplace transformation $\sigma_z(t)=\mathsf{L}^{-1}\mathsf{L}\left\{\sigma_z\right\}$ of the previous expression (32) and move from the complex domain of Laplace transform to the time domain.

That, it is necessary to develop the expression $\mathbf{E}_{\alpha}\left\{\mathsf{D}_t^{\alpha}\left[\varepsilon_z\right]_{z,,0}\right\}\frac{1}{p}\cdot\frac{1}{\left\langle 1+\left(\frac{\mathbf{E}_{\alpha}}{\mathbf{E}_e}+\frac{\mathbf{E}_{\alpha}}{\mathbf{E}}\right)p^{\alpha}\right\rangle}$ in order by powers of p, which is a complex number, using the formula (6) as well as that is $\left(\frac{\mathbf{E}_{\alpha}}{\mathbf{E}_e}+\frac{\mathbf{E}_{\alpha}}{\mathbf{E}}\right)p^{\alpha}<<1$, and we can write:

$$\mathsf{L}\{\sigma_z\} = \approx \mathbf{E}_\alpha \left\{\mathsf{D}_t^\alpha\left[\varepsilon_z\right]_{z,,0}\right\}\frac{1}{p}\cdot\left\langle 1+\sum_{k=1}^{\infty}(-1)^k\left(\frac{\mathbf{E}_\alpha}{\mathbf{E}_e}+\frac{\mathbf{E}_\alpha}{\mathbf{E}}\right)^k p^{k\alpha}\right\rangle \tag{33}$$

because

$$\frac{1}{1+\dfrac{\mathbf{E}_\alpha}{\mathbf{E}+\mathbf{E}_e}p^{\varepsilon}} \approx \left\langle 1+1\sum_{k=1}^{\infty}(-1)^k\left(\frac{\mathbf{E}_\alpha}{(\mathbf{E}+\mathbf{E}_e)}\right)^k p^{k\alpha}\right\rangle$$

The inverse Laplace transform $\sigma_z(t) = \mathsf{L}^{-1}\mathsf{L}\{\sigma_z\}$ of the previous expression $\mathsf{L}\{\sigma_z\}$, (33), now gives an approximate analytical expression for the normal stress $\sigma_z(t)$ in the time domain, in the standard light rheological complex Maxwell-Faraday model, of the fractional type, with the property of piezoelectricity, in the form of a power-order by degrees of time, during model is under constant axial dilatation during time period when model is under constant dilatation $\left\{\mathsf{D}_t^\alpha\left[\varepsilon_z\right]_{z,,0}\right\} = const$, in the form of:

$$\sigma_z(t) = \mathsf{L}^{-1}\mathsf{L}\{\sigma_z\} \approx \mathbf{E}_\alpha \left\{\mathsf{D}_t^\alpha\left[\varepsilon_z\right]_{z,,0}\right\}\cdot\left\{1+\sum_{k=1}^{\infty}(-1)^k\left(\frac{\mathbf{E}_\alpha}{\mathbf{E}_e}+\frac{\mathbf{E}_\alpha}{\mathbf{E}}p\right)^k \frac{t^{(2-\alpha)k+1}}{\Gamma(2k+2-\alpha k)}\right\} \tag{34}$$

$$\sigma_z(t) \approx \mathbf{E}_\alpha \left\{\mathsf{D}_t^\alpha\left[\varepsilon_z\right]_{z,,0}\right\}\cdot\left\{1+\sum_{k=1}^{\infty}(-1)^k\left(\frac{\mathbf{E}_\alpha}{\mathbf{E}_e}+\frac{\mathbf{E}_\alpha}{\mathbf{E}}p\right)^k \frac{t^{(2-\alpha)k+1}}{\Gamma(2k+2-\alpha k)}\right\}$$

The previous expression (34) describes the temporal behavior of the normal stress $\sigma_z(t)$ model of the standard light rheological complex Maxwell-Faraday model, of the fractional type, with the property of piezoelectricity, during the time period when the model is under constant dilatation $\left\{\mathsf{D}_t^\alpha\left[\varepsilon_z\right]_{z,,0}\right\} = const$. We see that with time that normal stress $\sigma_z(t)$ decreases and tends to zero, and this behavior of the model is called normal stress relaxation.

Electric voltage of the polarization field:

Electric voltage polarization fields in Faraday's element of the standard light rheological complex Maxwell-Faraday model, of the fractional type, with the property of piezoelectricity, during time period when model is under constant dilatation $\left\{\mathsf{D}_t^\alpha\left[\varepsilon_z\right]_{z,,0}\right\} = const$, is:

$$\mathbf{V}_z(t) = -g\sigma_z(t) \approx -g\mathbf{E}_\alpha \left\{\mathsf{D}_t^\alpha\left[\varepsilon_z\right]_{z,,0}\right\}\cdot\left\{1+\sum_{k=1}^{\infty}(-1)^k\left(\frac{\mathbf{E}_\alpha}{\mathbf{E}_e}+\frac{\mathbf{E}_\alpha}{\mathbf{E}}p\right)^k \frac{t^{(2-\alpha)k+1}}{\Gamma(2k+2-\alpha k)}\right\} \tag{35}$$

$$\mathbf{V}_z(t) \approx -g\mathbf{E}_\alpha \left\{\mathsf{D}_t^\alpha\left[\varepsilon_z\right]_{z,,0}\right\}\cdot\left\{1+\sum_{k=1}^{\infty}(-1)^k\left(\frac{\mathbf{E}_\alpha}{\mathbf{E}_e}+\frac{\mathbf{E}_\alpha}{\mathbf{E}}p\right)^k \frac{t^{(2-\alpha)k+1}}{\Gamma(2k+2-\alpha k)}\right\}$$

And dielectric displacement of polarization fields in Faraday's element of the standard light rheological complex Maxwell-Faraday model, of the fractional type, with the property of piezoelectricity, during time period when model is under constant dilatation $\left\{\mathsf{D}_t^{\alpha}\left[\varepsilon_z\right]_{z,,0}\right\}=const$, is:

$$D_z(t)=b\sigma_z(t)\approx b\mathbf{E}_{\alpha}\left\{\mathsf{D}_t^{\alpha}\left[\varepsilon_z\right]_{z,,0}\right\}\cdot\left\{1+\sum_{k=1}^{\infty}(-1)^k\left(\frac{\mathbf{E}_{\alpha}}{\mathbf{E}_e}+\frac{\mathbf{E}_{\alpha}}{\mathbf{E}}p\right)^k\frac{t^{(2-\alpha)k+1}}{\Gamma(2k+2-\alpha k)}\right\} \tag{36}$$

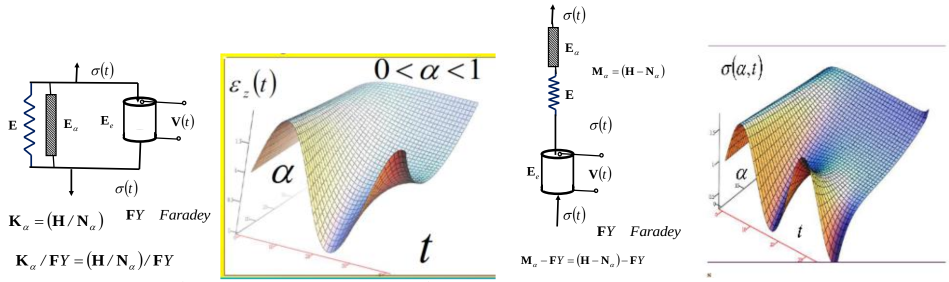
We can see from the previous solution (34), and from the graph of the relaxation surface of the normal stress $\sigma_z(t)$, that in that case the normal stress will asymptotically decrease and tend to zero, as shown in Fig. 3. The occurrence of a decrease in normal stress $\sigma_z(t)$ with the passage of time at constant velocity of axial dilation, fractional type $\left\{\mathsf{D}_t^{\alpha}\left[\varepsilon_z\right]_{z,,0}\right\}=const$ is called normal stress $\sigma_z(t)$ relax**a**tion, fractional type in model of the standard light rheological complex Maxwell-Faraday model, of the fractional type, with the property of piezoelectricity.

From expressions (34) and (36) we see that in the model of the standard light rheological complex Maxwell-Faraday model, of the fractional type, with the property of piezoelectricity, and polarization, and both electric voltage and dielectric displacement express the properties of polarization relaxation, relaxation of electric stress and relaxation of dielectric displacement over time, when the model is subjected to a constant rate of fractional type axial dilation $\left\{\mathsf{D}_t^{\alpha}\left[\varepsilon_z\right]_{z,,0}\right\}=const$.

For Concluding this part of this manuscript, in Fig. 3**,** the structure of two basic rheological standard light complex models of fractional type and piezoelectriv property and contributions of the properties in form of dilatation surfaces of subsequent elasticity, i.e. normal stress relaxation surface is presented.

Fig. 3.a shows the structure of basic rheological standard light complex Kelvin-Voigt-Faraday's model of fractional type with piezoelectric property, which contains in its structure parallel connected basic elements: Hooke's ideal elastic element and basic Newton's ideal viscous element of the fractional type and Faraday's ideal elastic and piezoelectric element. Fig. 3.b shows space surface of the subsequent elasticity (or subsequent electric current) of the basic rheological standard light complex Kelvin-Voigt-Faraday's model of the fractional type in coordinate system axial dilatation (or electric current), exponent of fractional order differentiation and time.

Fig. 3.c shows the structure of the basic rheological standard light complex Maxwell-Faraday's model of the fractional type with piezoelectric property, which contains in its structure, connected in series: the basic Hooke's ideally elastic element and the basic Newton's ideally viscous element of the fractional type and Faraday's ideal elastic and piezoelectric element. Fig. 3.d shows the normal stress relaxation surface (or electric voltage relaxation surface) of the basic rheological standard light complex Maxwell-Faraday's model of the fractional type, in coordinate system normal stress (electric voltage), exponent of fractional order differentiation and time.



$$\varepsilon_z(t) = \mathsf{L}^{-1}\mathsf{L}\{\varepsilon_z\} = \frac{\sigma_{z,0}}{(\mathbf{E}+\mathbf{E}_e)} \cdot \left\langle 1 + \sum_{k=1}^{\infty} (-1)^k \left(\frac{\mathbf{E}_\alpha}{(\mathbf{E}+\mathbf{E}_e)} \right)^k \frac{t^{(2-\alpha)k+1}}{\Gamma(2k+2-\alpha k)} \right\rangle$$

$$\sigma_z(t) = \mathsf{L}^{-1}\mathsf{L}\{\sigma_z\} \approx \mathbf{E}_\alpha \left\{ \mathsf{D}_t^\alpha [\varepsilon_z]_{z,,0} \right\} \cdot \left\{ 1 + \sum_{k=1}^{\infty} (-1)^k \left(\frac{\mathbf{E}_\alpha}{\mathbf{E}_e} + \frac{\mathbf{E}_\alpha}{\mathbf{E}} p \right)^k \frac{t^{(2-\alpha)k+1}}{\Gamma(2k+2-\alpha k)} \right\}$$

a b c d

Fig. 3. The structure of two basic rheological standard light complex models of fractional type and piezoelectriv property and contributions of the properties in form of dilatation surfaces of subsequent elasticity, i.e. normal stress relaxation surface

3. TWO RHEOLOGICAL DYNAMICAL SYSTEMS FRACTIONAL TYPE WITH PIEZOELECTRICAL PROPERTY

In the previous chapter 2, we defined two standard light rheological complex base models, which we can use as connection models between rigid bodies in discrete systems with one or more degrees of freedom of movement. We call such discrete systems, in which the connection between rigid bodies is achieved by a single or a larger number of standard light rheological models, discrete rheologic dynamic systems. In this chapter, through two subchapters, we will theoretically study the dynamics of two models of rheological discrete dynamic systems in which a rigid body, which moves translationally is bound by one of the two, previously studied in chapter 2, standard light rheological basic complex models. We call those two rheological dynamic systems: First, rheological Kelvin-Voigt-Faraday's dynamic system of fractional type with piezoelectric property and show that it is a rheological oscillator, fractional type with piezoelectric property and with one degree of freedom of oscillation; Second, a rheological Maxwell-Faraday dynamic system of fractional type with piezoelectric property and show that it is a rheological creeper-crawler with flow, fractional type with piezoelectric property and with one external and two internal degrees of freedom of movement-creep; Dynamics and motion of these rheological dynamic systems and different, but in both cases their dynamics are described first by one non-homogeneous differential equation of fractional order, and second by a system of three non-homogeneous fractional-type differential equations.

3.1. The rheological Kelvin-Voigt-Faraday’s dynamical systems fractional type with piezoectrical property

Rheological Kelvin-Voigt-Faraday's dynamic system of fractional type with piezoelectric property, shown in the left part of Figure 4, consists of two material points (two rigid bodies) that move translationally on an ideally smooth plane, and are connected to each other by standard light basic rheological complex Kelvin-Voigt-Faraday's model, whose structure is shown on the right part of Figure

Material points are mass m_1 and m_2. The stiffnesses of the standard light basic rheological complex Kelvin-Voigt-Faraday's model are c_0, c_α and c_e. This rheological system has two degrees of freedom of movement, but only one degree of freedom of oscillation, one material point in relation to another, whose positions are determined by the generalized coordinates x_1 and x_2.

The observed rheological Kelvin-Voigt-Faraday dynamic system is subjected to the action of a single-frequency periodic force $F(t) = F\ _0 sin(\Omega t + \phi_0)$ acting through another material point, as shown in Fig. 4.

The system of ordinary inhomogeneous differential equations of the fractional order of the dynamics of the system from Figure 6, when the system is acted upon by an active force $F(t) = F_0 sin(\Omega t + \varphi_0)$, is (see Refs [24, 34-37]):

$$m_1\ddot{x}_1(t) - c[x_2(t) - x_1(t)] - c_\alpha \mathsf{D}_t^\alpha [x_2(t) - x_1(t)] = 0 \tag{37}$$

$$m_2\ddot{x}_2(t) + c[x_2(t) - x_1(t)] + c_\alpha \mathsf{D}_t^\alpha [x_2(t) - x_1(t)] = F(t) = F_0 sin(\Omega t + \varphi_0) \tag{38}$$

We introduce the following notations:

$$\omega_0^2 = \frac{(c_0 + c_e)(m_1 + m_2)}{m_1 m_2},\ \omega_\alpha^2 = \frac{c_\alpha (m_1 + m_2)}{m_1 m_2},\ c = c_0 + c_e \ \text{и}\ h = \frac{m_1}{m_1 m_2} \tag{39}$$

and introduce a new independent generalized coordinate $x(t) = [x_2(t) - x_1(t)]$, which represents the relative displacement of one material point (rigid body in translator motion) in relation to another.

The system of ordinary inhomogeneous differential equations of the fractional order of the dynamics of the system from Figure 6, when the system is acted upon by an active force $F(t) = F_0 sin(\Omega t + \varphi_0)$, is (see References [24, 34-37]):

$$m_1\ddot{x}_1(t) - c[x_2(t) - x_1(t)] - c_\alpha \mathsf{D}_t^\alpha [x_2(t) - x_1(t)] = 0 \tag{37}$$

$$m_2\ddot{x}_2(t) + c[x_2(t) - x_1(t)] + c_\alpha \mathsf{D}_t^\alpha [x_2(t) - x_1(t)] = F(t) = F_0 sin(\Omega t + \varphi_0) \tag{38}$$

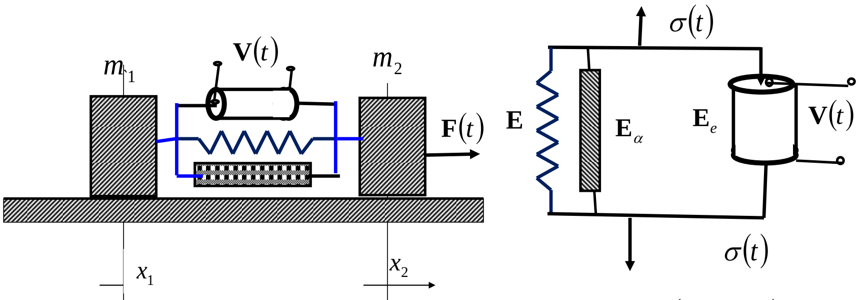
We introduce the following notations:

$$\omega_0^2 = \frac{(c_0 + c_e)(m_1 + m_2)}{m_1 m_2},\ \omega_\alpha^2 = \frac{c_\alpha (m_1 + m_2)}{m_1 m_2},\ c = c_0 + c_e \ \text{и}\ h = \frac{m_1}{m_1 m_2} \tag{39}$$

and introduce a new independent generalized coordinate $x(t) = [x_2(t) - x_1(t)]$, which represents the relative displacement of one material point (rigid body in translator motion) in relation to another.

If we now multiply the first inhomogeneous fractional differential equation (37) by m_2, and the second (38) by m_1, and then subtract the first from the second, we get an ordinary inhomogeneous fractional differential equation of the form:

$$\ddot{x}(t) + \omega_0^2 x(t) + \omega_\alpha^2 \mathsf{D}_t^\alpha [x(t)] = h\, sin(\Omega t + \varphi_0) \tag{40}$$



$$\mathbf{F}_{\mathbf{K}_\alpha} = -\left[c_0(x_2-x_1)+c_e(x_2-x_1)+c_\alpha \mathrm{D}_t^\alpha\left[(x_2-x_1)\right]\right] \qquad \mathbf{K}_\alpha = (\mathbf{H}/\mathbf{N}_\alpha)$$

$$\mathbf{F}(t) = \mathbf{F}_0 sim(\Omega t+\varphi_0) \qquad \mathbf{K}/\mathbf{FY}_\alpha = (\mathbf{H}/\mathbf{N}_\alpha)/\mathbf{FY}$$

$$\mathbf{V}(t) = g\sigma_z(t)$$

$$\mathbf{E}_k = \frac{1}{2}\sum_{i=1}^{i=2} m_i \dot{x}_i^2 \quad \mathbf{E}_p = \frac{1}{2}(c_0+c_e)\left[(x_2-x_1)^2\right] \quad \mathbf{\Phi}_w = \frac{1}{2}c_\alpha \mathrm{D}_t^\alpha\left[(x_2-x_1)^2\right]$$

Fig. 4 Fractional-type rheological Kelvin-Voigt-Faraday discrete dynamic system with piezoelectric property (left) and the structure of a standard light basic rheological complex Kelvin-Voigt-Faraday bond model (right)

And if we add the differential equations of fractional order (37) and (38), we get one ordinary differential equation of the form:

$$m_2\ddot{x}_2(t) + m_1\ddot{x}_1(t) = F(t) = F_0 sin(\Omega t + \varphi_0) \tag{41}$$

which by integration gives

$$K = m_2\dot{x}_2(t) + m_1\dot{x}_1(t) = -\frac{F_0}{\Omega}cos(\Omega t + \varphi_0) + C_0 \tag{42}$$

from which we conclude that the entire system, both material points (both rigid bodies) move harmonically with an oscillatory amount of motion (harmonic oscillatory impulse).

However, the inhomogeneous ordinary differential equation of fractional order (40) indicates that it is a rheological dynamic oscillator of fractional order rheological Kelvin-Voigt-Faraday's dynamic system, whose coefficients (39) depend on the stiffness c_0, c_α and c_e. All three rheological elements, Hooke's ideally elastic element, Newton's viscous fractional type element and Faraday's ideally elastic and piezoelectric elements element, connected in parallel.

The solution of this inhomogeneous differential equation of fractional order (40) can be found in detail in Refs [22-24, 33], so we will not dwell on the details of the solution, but will determine and list only the solutions and expressions whose analysis we will use to explain the specifics of the dynamics of the rheological Kelvin - Voigt-Faraday's dynamic system - rheological oscillator, which is subjected to the effect of a single-frequency periodic force $F(t) = F_0 sin(\Omega t + \varphi_0)$.

The Laplace transformation applied to an inhomogeneous differential equation of fractional order (40) gives one algebraic equation per unknown Laplace transformation $\mathsf{L}\{x(t)\}$ of the unknown independent generalized coordinate of the rheological Kelvin-Voigt-Faraday's dynamic system.

Now, the Laplace transformation $\mathsf{L}\{x(t)\}$ of the unknown independent generalized coordinate $x(t)$ of the rheological Kelvin-Voigt-Faraday’s dynamic system, of the fractional type with piezoelectric property is in the form (for detail see Refs [22, 23, 24])

$$\mathsf{L}\{x\} = \frac{px_0 + \dot{x}_0}{p^2 + \omega_0^2 + \omega_\alpha^2 p^\alpha} + \frac{1}{p^2 + \omega_0^2 + \omega_\alpha^2 p^\alpha} \frac{\Omega}{p^2 + \Omega^2} \tag{43}$$

Now, we can divide this expression (42) for the Laplace transformation $\mathsf{L}\{x(t)\}$ of the independent generalized coordinate $x(t)$ into two parts, the part corresponding to its own free oscillations $\mathsf{L}\{x_{free}(t)\}$ of the fractional type and the part corresponding to the forced oscillations $\mathsf{L}\{x_{part}(t)\}$ of the fractional type under the action of a sinusoidal periodic force:

$$\mathsf{L}\{x(t)\} = \mathsf{L}\{x_{free}(t)\} + \mathsf{L}\{x_{part}(t)\} = \mathsf{L}\{x_{likesin}(t)\} + \mathsf{L}\{x_{likecos}(t)\} + \mathsf{L}\{x_{part}(t)\}$$

Then, the analytical approximate solution of the previous non-homogeneous ordinary differential equation, of fractional order (40), describing displacement of the dynamics of the rheological Kelvin-Voigt-Faraday’s dynamic system, of the fractional type with piezoelectric property, from the interval $\alpha \in (0.1)$ of fractional order differentiation, is of the form (for details see Refs [22, 23, 24, 35]):

$$\begin{aligned} x(t) = {} & \sum_{k=0}^{\infty} (-1)^k \omega_\alpha^{2k} t^{2k} \sum_{j=0}^{k} \binom{k}{j} \frac{\omega_\alpha^{2j} t^{-\alpha j}}{\omega_o^{2j}} \left[\frac{x_0}{\Gamma(2k+1-\alpha j)} + \frac{\dot{x}_0 t}{\Gamma(2k+2-\alpha j)} \right] + \\ & + \int_0^t \langle h\, sin(\Omega(t-\tau)) + \varphi_0 \rangle \left\langle \sum_{k=0}^{\infty} (-1)^k \omega_\alpha^{2k} \tau^{2k+1} \sum_{m=0}^{k} \binom{k}{m} \frac{\omega_\alpha^{-2m} \tau^{-\alpha m}}{\omega_0^{2m} \Gamma(2k+2-\alpha m)} \right\rangle d\tau \\ & \alpha \in (0.1) \end{aligned} \tag{44}$$

Modes of eigen rheological oscillations are:

$$x_{free}(t) = \mathsf{L}^{-1}\mathsf{L}\{x_{free}(t)\}\backslash = \sum_{k=0}^{\infty} (-1)^k \omega_\alpha^{2k} t^{2k} \sum_{j=0}^{k} \binom{k}{j} \frac{\omega_\alpha^{2j} t^{-\alpha j}}{\omega_o^{2j}} \left[\frac{x_0}{\Gamma(2k+1-\alpha j)} + \frac{\dot{x}_0 t}{\Gamma(2k+2-\alpha j)} \right] \tag{45}$$

$$x_{likecos}(t) = \sum_{k=0}^{\infty} (-1)^k \omega_\alpha^{2k} t^{2k} \sum_{j=0}^{k} \binom{k}{j} \frac{\omega_\alpha^{2j} t^{-\alpha j}}{\omega_o^{2j}} \left[\frac{x_0}{\Gamma(2k+1-\alpha j)} \backslash \right] \tag{46}$$

$$x_{likesin}(t) = \sum_{k=0}^{\infty} (-1)^k \omega_\alpha^{2k} t^{2k} \sum_{j=0}^{k} \binom{k}{j} \frac{\omega_\alpha^{2j} t^{-\alpha j}}{\omega_o^{2j}} \left[\frac{\dot{x}_0 t}{\Gamma(2k+2-\alpha j)} \right] \tag{47}$$

The axial dilatation of the parallel-connected basic Faraday's ideally elastic and piezoelectric element in the rheological Kelvin-Voigt-Faraday's dynamic system, of the fractional type with piezoelectric property, is:

$$\varepsilon_z(t)=\frac{x_2(t)-x_1(t)}{\ell_0}=\frac{x(t)}{\ell_0}$$

$$\begin{aligned}\varepsilon_z(t)=&\frac{1}{\ell_0}\sum_{k=0}^{\infty}(-1)^k\omega_\alpha^{2k}t^{2k}\sum_{j=0}^{k}\binom{k}{j}\frac{\omega_\alpha^{2j}t^{-\alpha j}}{\omega_o^{2j}}\left[\frac{x_0}{\Gamma(2k+1-\alpha j)}+\frac{\dot{x}_0t}{\Gamma(2k+2-\alpha j)}\right]+\\&+\frac{h}{\ell_0}\int_0^t\langle sin(\Omega_0(t-\tau))\rangle\left\langle\sum_{k=0}^{\infty}(-1)^k\omega_\alpha^{2k}\tau^{2k+1}\sum_{j=0}^{k}\binom{k}{j}\frac{\omega_\alpha^{-2m}\tau^{-\alpha j}}{\omega_0^{2m}\Gamma(2k+2-\alpha j)}\right\rangle d\tau\end{aligned}\tag{48}$$

Mechanical normal stress in parallel connected basic Faraday piezoelectric element in rheological Kelvin-Voigt-Faraday's dynamic system, of the fractional type with piezoelectric property, is:

$$\begin{aligned}\sigma_z(t)=\mathbf{E_e}\varepsilon_z(t)=\mathbf{E_e}\frac{x(t)}{\ell_0}=&\frac{\mathbf{E_e}}{\ell_0}\sum_{k=0}^{\infty}(-1)^k\omega_\alpha^{2k}t^{2k}\sum_{j=0}^{k}\binom{k}{j}\frac{\omega_\alpha^{2j}t^{-\alpha j}}{\omega_o^{2j}}\left[\frac{x_0}{\Gamma(2k+1-\alpha j)}+\frac{\dot{x}_0t}{\Gamma(2k+2-\alpha j)}\right]+\\&+\frac{\mathbf{E_e}h}{\ell_0}\int_0^t\langle sin(\Omega_0(t-\tau))\rangle\left\langle\sum_{k=0}^{\infty}(-1)^k\omega_\alpha^{2k}\tau^{2k+1}\sum_{j=0}^{k}\binom{k}{j}\frac{\omega_\alpha^{-2m}\tau^{-\alpha j}}{\omega_0^{2m}\Gamma(2k+2-\alpha j)}\right\rangle d\tau\end{aligned}\tag{49}$$

The electric voltage of the electric polarization in the parallel connected basic Faraday piezoelectric element in the rheological Kelvin-Voigt-Faraday's dynamic system, of the fractional type with piezoelectric property, is:

$$\begin{aligned}\mathbf{V}_z(t)=-g\sigma_z(t)=&\frac{\mathbf{E_e}g}{\ell_0}\sum_{k=0}^{\infty}(-1)^k\omega_\alpha^{2k}t^{2k}\sum_{j=0}^{k}\binom{k}{j}\frac{\omega_\alpha^{2j}t^{-\alpha j}}{\omega_o^{2j}}\left[\frac{x_0}{\Gamma(2k+1-\alpha j)}+\frac{\dot{x}_0t}{\Gamma(2k+2-\alpha j)}\right]+\\&+\frac{\mathbf{E_e}hg}{\ell_0}\int_0^t\langle sin(\Omega_0(t-\tau))\rangle\left\langle\sum_{k=0}^{\infty}(-1)^k\omega_\alpha^{2k}\tau^{2k+1}\sum_{j=0}^{k}\binom{k}{j}\frac{\omega_\alpha^{-2m}\tau^{-\alpha j}}{\omega_0^{2m}\Gamma(2k+2-\alpha j)}\right\rangle d\tau\end{aligned}\tag{50}$$

Dielectric displacement in the field of electric polarization in the parallel-connected basic Faraday piezoelectric element in the rheological Kelvin-Voigt-Faraday's dynamic system, of the fractional type with piezoelectric property, is:

$$\begin{aligned}D_z(t)=b\sigma_z(t)=&\frac{\mathbf{E_e}b}{\ell_0}\sum_{k=0}^{\infty}(-1)^k\omega_\alpha^{2k}t^{2k}\sum_{j=0}^{k}\binom{k}{j}\frac{\omega_\alpha^{2j}t^{-\alpha j}}{\omega_o^{2j}}\left[\frac{x_0}{\Gamma(2k+1-\alpha j)}+\frac{\dot{x}_0t}{\Gamma(2k+2-\alpha j)}\right]+\\&+\frac{\mathbf{E_e}hb}{\ell_0}\int_0^t\langle sin(\Omega_0(t-\tau))\rangle\left\langle\sum_{k=0}^{\infty}(-1)^k\omega_\alpha^{2k}\tau^{2k+1}\sum_{j=0}^{k}\binom{k}{j}\frac{\omega_\alpha^{-2m}\tau^{-\alpha j}}{\omega_0^{2m}\Gamma(2k+2-\alpha j)}\right\rangle d\tau\end{aligned}\tag{51}$$

and

$$\begin{aligned}D_z(t)=e\varepsilon_z(t)=&\frac{e}{\ell_0}\sum_{k=0}^{\infty}(-1)^k\omega_\alpha^{2k}t^{2k}\sum_{j=0}^{k}\binom{k}{j}\frac{\omega_\alpha^{2j}t^{-\alpha j}}{\omega_o^{2j}}\left[\frac{x_0}{\Gamma(2k+1-\alpha j)}+\frac{\dot{x}_0t}{\Gamma(2k+2-\alpha j)}\right]+\\&+\frac{eh}{\ell_0}\int_0^t\langle sin(\Omega_0(t-\tau))\rangle\left\langle\sum_{k=0}^{\infty}(-1)^k\omega_\alpha^{2k}\tau^{2k+1}\sum_{j=0}^{k}\binom{k}{j}\frac{\omega_\alpha^{-2m}\tau^{-\alpha j}}{\omega_0^{2m}\Gamma(2k+2-\alpha j)}\right\rangle d\tau\end{aligned}\tag{52}$$

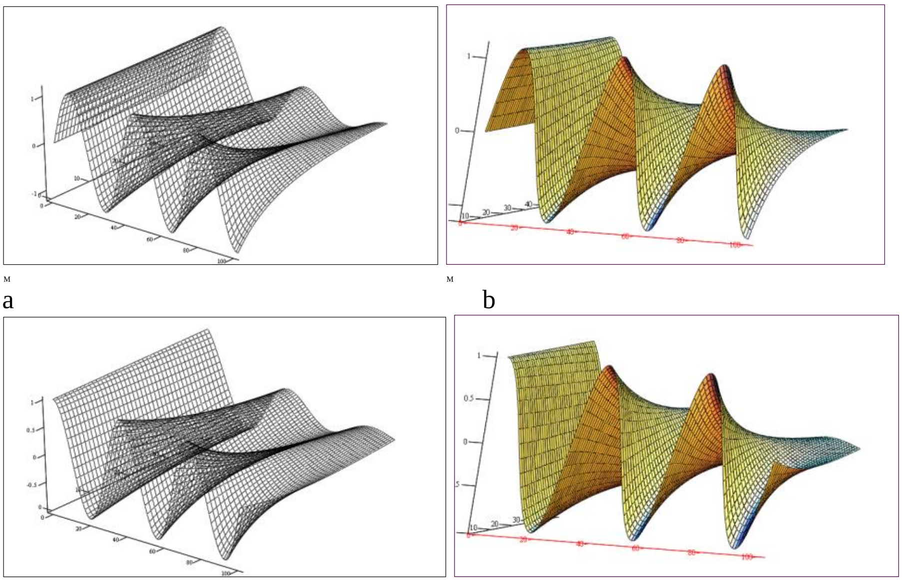
The basic modes of dilatation, mechanical normal stress, electric polarization voltage and dielectric displacement are attached to their own, similar to cosines and similar to sinuses, fractional-type modes and forced modes, fractional-type piezoelectric properties, dissipative properties..

3.1.1. The graphical presentation of some characteristic properties of dynamics of the rheological Kelvin-Voigt-Faraday's dynamical systems fractional type with piezoectrical property

In this sub-chapter, first, we will study and show graphically in the form of spatial surfaces some properties and shapes of eigenmodes and forced modes of dynamics of the rheological Kelvin-Voigt-Faraday oscillator, fractional type and with piezoelectric property, and in coordinate systems with coordinate axes: elongation of eigenmode or elongation of forced mode, exponent of the differential operator of fractional order differentiation in the interval $0<\alpha<1$ and time.

Secondly, we will, also, study some properties of the Laplace transforms of the eigenmodes and forced modes of dynamics of the rheological Kelvin-Voigt-Faraday oscillator, fractional type and with piezoelectric properties.

Fig. 5 shown graphical set of space surfaces of characteristic eigenmodes, in the coordinate system of the coordinate axes: elongation of eigenmode, exponent of the differential operator of fractional differentiation in the interval $0<\alpha<1$ and time. a) and b) eigenmode "like sin" and c) and d) eigenmode "like cos" mode for different parameters of dynamics of the rheological Kelvin-Voigt-Faraday's dynamical systems fractional type with piezoelectric property

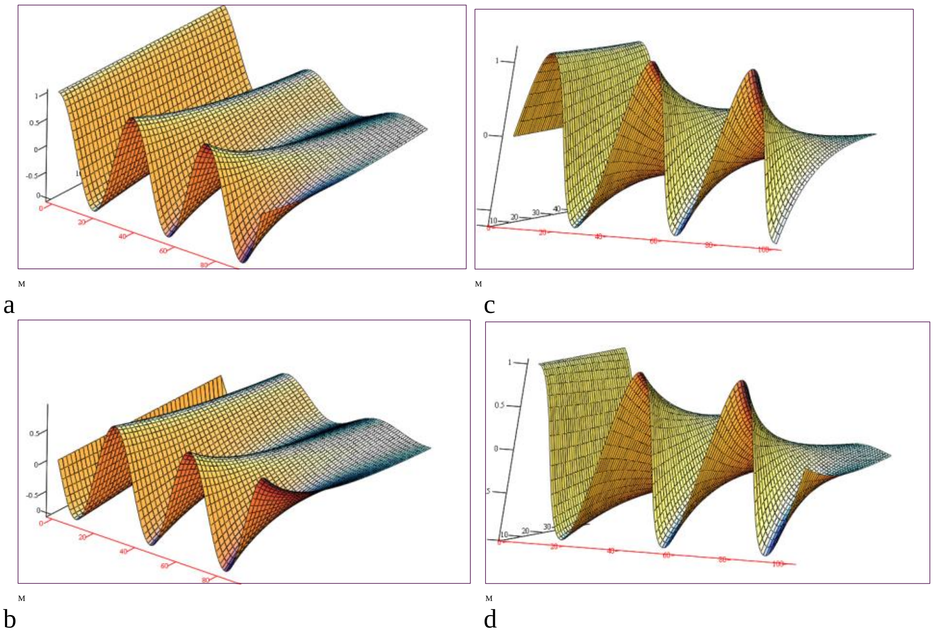


a b

c d

Figure 5. Graphical set of space surfaces of characteristic eigenmodes, in the coordinate system of the coordinate axes: elongation of eigenmode, exponent of the differential operatorof fractional order differentiation in the interval $0<\alpha<1\ 0<\alpha<1$ and time

Fig. 6 shown graphical set of space surfaces of characteristic eigenmodes and their first derivatives in time, in the coordinate system of coordinate axes: elongation of eigenmode or elongation of their first derivative in time, exponent of the differential operator of fractional order differentiation in the interval $0 < \alpha < 1$ and time. , a) eigenmode "like cos" , b) the derivative of the eigenmode "like cos" (in the form minus "likesin"), c) the eigenmode "likesin" and d) the derivative of the eigenmode "likesin" (in the form "like cos") for the corresponding parameters of dynamics of the rheological Kelvin-Voigt-Faraday's dynamical systems fractional type with piezoelectric property. Figs 7 and 8 present graphical representations of the spatial surfaces corresponding to particular solutions of an inhomogeneous fractional-order ordinary differential equation. These solutions describe the elongations of fractional-type forced modes for a rheological oscillator with one degree of freedom, governing the forced oscillatory dynamics of the fractional-order Kelvin–Voigt–Faraday rheological dynamical system with piezoelectric properties.

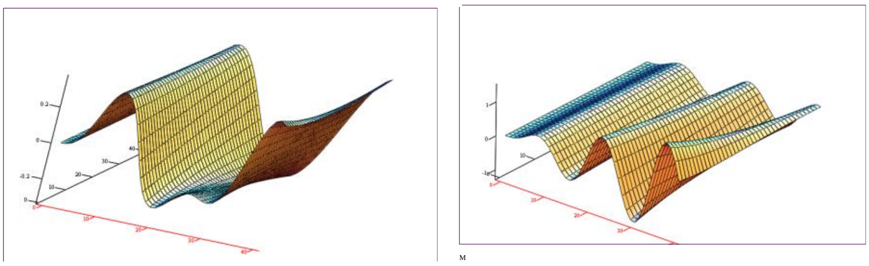


a c

b d

Fig. 6. Graphical set of space surfaces of characteristic eigenmodes and their first derivatives in time, in the coordinate system of coordinate axes: elongation of eigenmode or elongation of their first derivative in time, exponent of the differential operator of fractional order differentiation in the interval $0 < \alpha < 1$ and time

The spatial surfaces are depicted in a three-dimensional coordinate system defined by the elongation of the forced mode, time, and the fractional-order differentiation exponent, within the interval $\alpha \in (0\,,1)$. The results are shown for different values of the circular excitation frequency. In particular, the cases $\Omega_0 = 1$, $\Omega_0 = 2$ and $\Omega_0 = 10$ are illustrated.



a b

Fig. 7. Graphical representation of the space surface of the particular solution of the ordinary inhomogeneous differential equation, fractional order, which describes the elongation $x_{prtt}(t)$ of the forced mode, fractional type, for an rheological oscillator with one degree of freedom of forced oscillations, fractional type, of dynamics of the rheological Kelvin-Voigt-Faraday's dynamical systems fractional type with piezoelectric property, in the coordinate system: elongation $x_{prtt}(t)$ of the forced mode, time and exponent α fractional order differentiation in the interval $0 < \alpha < 1$, , and for the circular frequency in a) $\Omega_0 = 1$ and in b) $\Omega_0 = 2$

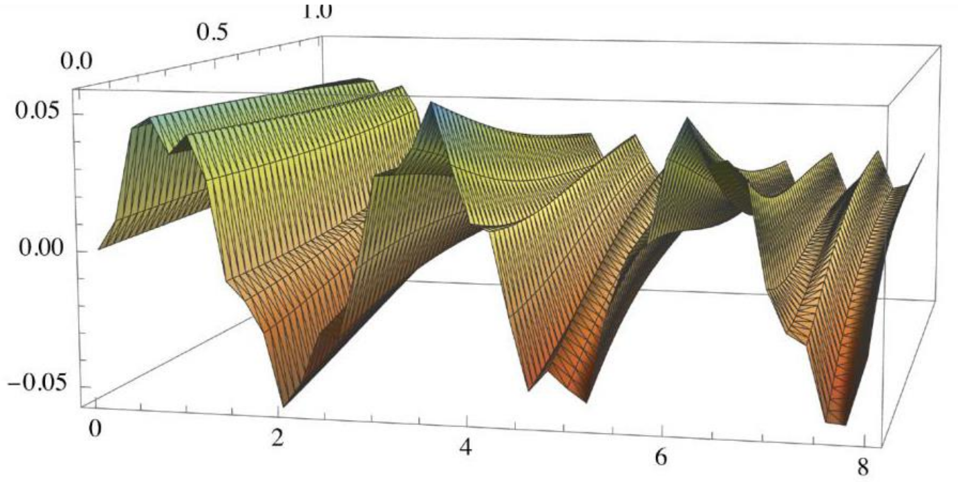


Fig. 8. Graphical representation of the space surface of the particular solution of the ordinary inhomogeneous differential equation, fractional order, which describes the elongation $x_{prtt}(t)$ of the forced mode, fractional type, for a rheological oscillator with one degree of freedom of forced oscillations, fractional type, of dynamics of the rheological Kelvin-Voigt-Faraday's dynamical systems fractional type with piezoelectric property, in the coordinate system: elongation $x_{prtt}(t)$ of the forced mode, time and exponent α fractional order differentiation in the interval $0 < \alpha < 1$, , and for the circular frequency $\Omega_0 = 10$

It can be observed that, for the selected parameters of the fractional-order oscillatory system with one degree of freedom under forced excitation, the dynamics of the fractional Kelvin–Voigt–Faraday rheological dynamical system with piezoelectric properties exhibits a characteristic response. In particular, for frequency values around $\Omega_0 = 0.5$, the particular solution displays a distinctive shape when compared with the cases corresponding to lower and higher excitation frequencies over the same time interval. Within this interval, excitation frequencies close to $\Omega_0 = 0.5$result in the smallest number of complete oscillations, whereas both lower and higher frequencies of external forcing applied to the mechanical and electrical oscillators produce a larger number of complete oscillation cycles.

Finally, it is emphasized that all spatial surface plots representing the natural and forced modes of the dynamics of the fractional Kelvin–Voigt–Faraday rheological oscillator with piezoelectric properties also qualitatively characterize the corresponding natural and forced modes of electric polarization of the Faraday rheological piezoelectric element. These include the electric voltage and dielectric displacement, as well as the associated mechanical stress and dilatation, which are therefore not presented separately.

As noted in the introduction to this subsection, we also investigate certain properties of the Laplace transforms of the eigenmodes and forced modes of the dynamics of the fractional Kelvin–Voigt–Faraday rheological oscillator with piezoelectric properties. These properties are illustrated by spatial surfaces represented in coordinate systems whose axes correspond to the Laplace transform of the elongation, the fractional-order differentiation exponent αin the interval $0 < \alpha < 1$, and the Laplace transform parameter p.

So let's form a set of Laplace transforms of eigenmodes and forced modes of oscillation dynamics of a rheological Kelvin-Voigt-Faraday rheological oscillator, fractional type with piezoelectric property.

The set of Laplace transforms $\mathsf{L}\left\{x_{free,likecos}\right\}$ and $\mathsf{L}\left\{x_{free,likesin}\right\}$ of the eigenmodes of the oscillation dynamics of the rheological Kelvin-Voigt-Faraday rheological oscillator, fractional type with piezoelectric property is:

$$\mathsf{L}\left\{x_{free,likecos}\right\}=\frac{p}{p^2+\left(\omega_0^2+\omega_e^2\right)+\omega_\alpha^2 p^\alpha} \tag{53}$$

$$\mathsf{L}\left\{x_{free,likesin}\right\}=\frac{1}{p^2+\left(\omega_0^2+\omega_e^2\right)+\omega_\alpha^2 p^\alpha} \tag{54}$$

The set of Laplace transforms $\mathsf{L}\left\{x_{Forced,sin}\right\}$ and $\mathsf{L}\left\{x_{Forced,cos}\right\}$ of the eigenmodes of the oscillation dynamics of the rheological Kelvin-Voigt-Faraday rheological oscillator, fractional type with piezoelectric property is:

$$\mathsf{L}\left\{x_{Forced,sin}\right\}=\frac{1}{p^2+\left(\omega_0^2+\omega_e^2\right)+\omega_\alpha^2 p^\alpha}\frac{\Omega}{p^2+\Omega^2} \tag{55}$$

$$\mathsf{L}\left\{x_{Forced,cos}\right\}=\frac{1}{p^2+\left(\omega_0^2+\omega_e^2\right)+\omega_\alpha^2 p^\alpha}\frac{p}{p^2+\Omega^2} \tag{56}$$

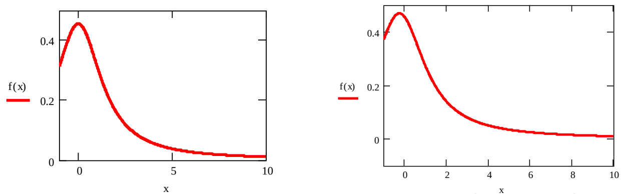
The following graphs show the contour lines of the Laplace transforms of the eigenmodes and forced modes of the fractional type with the piezoelectric property (Figs 9 and 10) for the dynamics of the rheological Kelvin-Voigt-Faraday's dynamic system, from Fig. 4, as well as the surfaces of the Laplace transforms of the eigenmodes and forced modes of the fractional type with the piezoelectric property in coordinate systems with Laplace transformation axes, fractional order differentiation exponent and Laplace transformation parameter p (Figs 11, 12).

3.2. The rheological Maxwell-Faraday’s dynamical systems fractional type with piezoectrical property

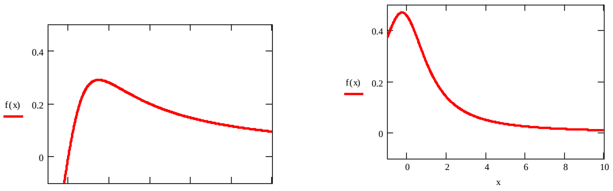
Two rheological Maxwell-Faraday's dynamic systems of fractional type with piezoelectric property, shown in the upper and lower parts of Figure 13, and both consist of one material point (one rigid body) moving translationally on an ideally smooth plane, and are bound by standard light basic rheological complex Maxwell-Faraday's model, whose structure is shown on the left part of Fig. 13.

The material points of both systems are mass m and its position is determined by an independent generalized coordinate x, which we will denote as the external independent generalized coordinate of the system.

The stiffnesses of the standard light basic rheological complex Maxwell-Faraday's model are c_0, c_α and c_e. This rheological system has three degrees of freedom of movement, but only one external degree of freedom of movement attached to that material point (rigid body) that moves translationally, and two internal degrees of freedom of movement that refer to the points of orderly-serially attachment of the rheological elements of Maxwell-Faraday's orderly connection model,



a- $\mathrm{L}\{x_{free,likesin}\}$, $\alpha = 0$ b- $\mathrm{L}\{x_{free,likesin}\}$, $\alpha = 1$



c- $\mathrm{L}\{x_{free,likeos}\}$, $\alpha = 0$ d- $\mathrm{L}\{x_{free,likeos}\}$, $\alpha = 1$

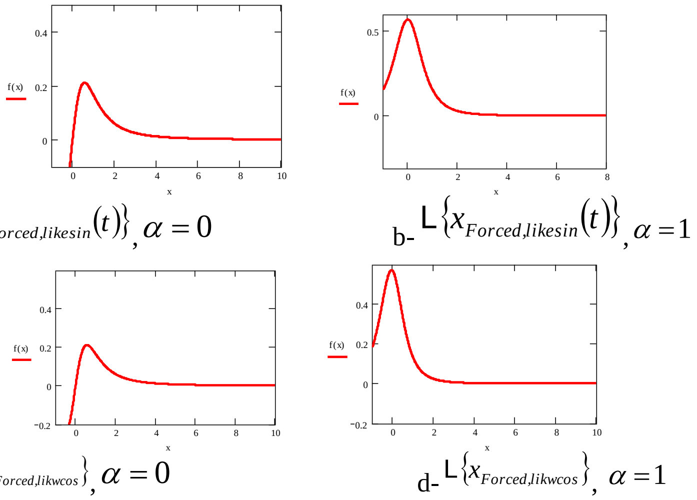
Fig. 9. Contour lines of space surfaces of Laplace transforms of eigenmodes, $\mathrm{L}\{x_{free,likesin}\}$ and $\mathrm{L}\{x_{free,likeos}\}$ and the parameter p of Laplace transforms, for the oscillatory dynamics of the rheological Kelvin-Voigt-Faraday’s dynamic system, from Fig. 4, drawn using analytical expressions (53) and (54) for the case when $\alpha = 0$, a and respectively $\alpha = 1$, b and d

The observed rheological Maxwell-Faraday's dynamic system is subjected to the effect of a single-frequency periodic force $F(t) = F_0 sin(\Omega t + \phi_0)$, as shown in Fig. 13.

The positions of the material point (rigid body) and points of order binding of the elements of the rheological standard light Maxwell-Faraday's model determined by independent generalized coordinates x, $x_{u,1}$ and $x_{u,2}$.

It means that unlike the rheological Kelvin-Voigt-Faraday dynamic system with one degree of freedom of oscillation, whose dynamics we studied in chapter 3.1, the rheological dynamic systems, shown in Figure 13, has one external degree of freedom of movement and two internal degrees of freedom movements, within the very basic complex standard light rheological Maxwell-Faraday's model, fractional type, with piezoelectric property, entered into the model by serial-order connections of structural elements.

Let us denote by $x(t)$ the independent generalized coordinate, which corresponds to the external degree of freedom of creep of material point (rigid body), and by $x_{u,1}(t)$ and $x_{u,2}(t)$ the internal independent generalized coordinate, which corresponds to the internal degrees of freedom of movement of the standard light rheological Maxwell-Faraday's model, fractional type, with piezoelectric property.



a- $\mathcal{L}\{x_{Forced,likesin}(t)\}$, $\alpha = 0$

b- $\mathcal{L}\{x_{Forced,likesin}(t)\}$, $\alpha = 1$

c- $\mathcal{L}\{x_{Forced,likwcos}\}$, $\alpha = 0$

d- $\mathcal{L}\{x_{Forced,likwcos}\}$, $\alpha = 1$

Fig. 10. Contour lines of the space surfaces of Laplace transforms of forced modes for the oscillatory dynamics of the rheological Kelvin-Voigt-Faraday's dynamic system , from Fig. 4, drawn using analytical expressions (55) and (56), and the cases of single-frequency periodic force action $F(t) = F_0 sin(\Omega t + \phi_0)$, i.e. , and $F(t) = F_0 cos(\Omega t + \phi_0)$, for the case when $\alpha = 0$, a and c, i.e. $\alpha = 1$, b and d

In the points of the internal series coupling of the simple elements of the structure of the standard light Maxwell-Faraday's model, of the fractional type, with piezoelectric property, let us place two fictitious material points of mass equal to zero, i.e., $m_{u,1} = 0$ and $m_{u,2} = 0$, while we set a system of differential equations of fractional order.

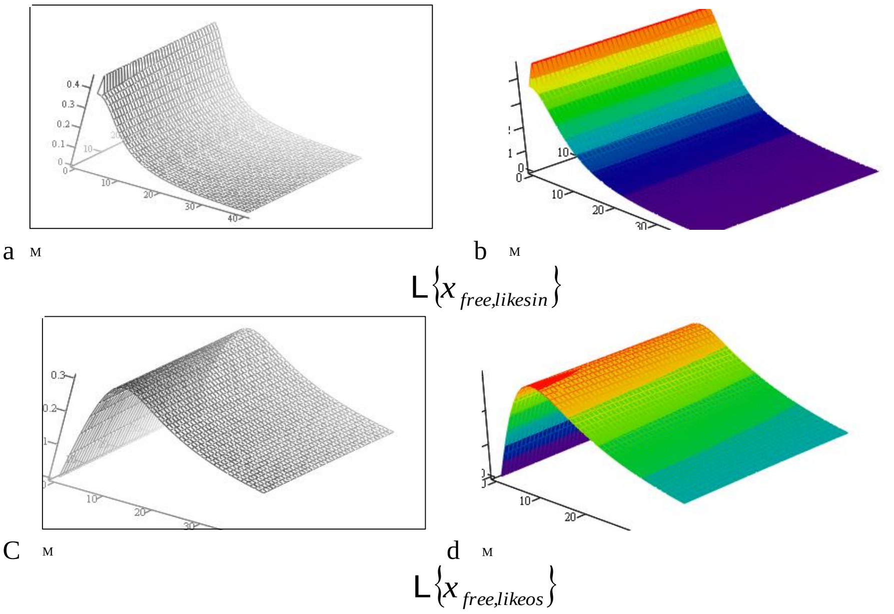


Fig. 11. Space surfaces of Laplace transforms of eigenmodes, and the Laplace transforms parameter p, for the oscillatory dynamics of the rheological Kelvin-Voigt-Faraday's dynamic system, from Fig. 4, drawn using analytical expressions (53) and (54)

That system of inhomogeneous ordinary differential equations of the fractional type, which describes the dynamics of the rheological Maxwell-Faraday's dynamic system, the model shown on the upper part of Fig. 13, is in the form:

$$m\ddot{x} + c_\alpha \mathrm{D}_t^\alpha\left[x(t) - x_{u,2}(t)\right] = F_0 sin(\Omega t + \varphi_0) \tag{57}$$

$$m_{u,1}\ddot{x}_{u,1} - c_0\left(x_{u,2}(t) - x_{u,1}(t)\right) + c_e x_{u,1}(t) = 0 \tag{58}$$

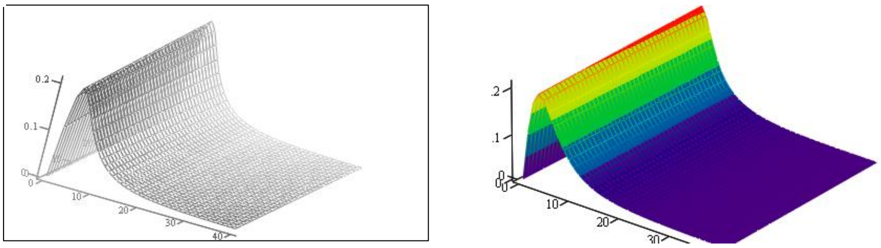
$$m_{u,2}\ddot{x}_{u,2} - c_\alpha \mathrm{D}_t^\alpha\left[x(t) - x_{u,2}(t)\right] + c_0\left(x_{u,2}(t) - x_{u,1}(t)\right) = 0 \tag{59}$$

For the second case, that system of inhomogeneous ordinary differential equations of the fractional type, which describes the dynamics of the rheological Maxwell-Faraday's dynamic system, the model shown in the lower part of Figure 13, is in the form:

$$m\ddot{x} + c_e\left(x(t) - x_{u,1}(t)\right) = F_0 sin(\Omega t + \varphi_0) \tag{60}$$

$$m_{u,1}\ddot{x}_{u,1} - c_e\left(x(t) - x_{u,1}(t)\right) - c_0\left(x_{u,2}(t) - x_{u,1}(t)\right) = 0 \tag{61}$$

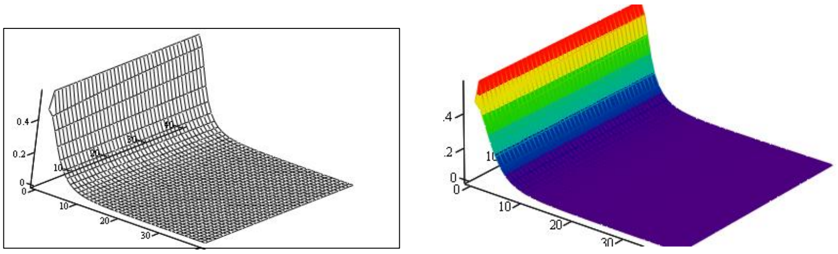
$$m_{u,2}\ddot{x}_{u,2} - c_\alpha \mathrm{D}_t^\alpha\left[x_{u,2}(t)\right] + c_0\left(x_{u,2}(t) - x_{u,1}(t)\right) = 0 \tag{62}$$



a M b M

$\mathrm{L}\left\{x_{Forced,likwcos}\right\}$

$M_{i,j} := f(x_i, y_j)$



C M d M

Fig. 12 Surfaces of Laplace transforms of forced modes, (a and b) $\mathrm{L}\left\{x_{Forced,likwcos}\right\}$ and (c and d) $\mathrm{L}\left\{x_{Forced,likesin}(t)\right\}$ fractional-type oscillations, for forced oscillatory dynamics of the rheological Kelvin-Voigt-Faraday's dynamic system from Figure 4, drawn using the analytical expression (56) and (55), and the cases of single-frequency periodic forces action, $F(t) = F_0 cos(\Omega t + \phi_0)$ and that is $F(t) = F_0 sin(\Omega t + \phi_0)$

Now let's introduce the following tags:

$$\omega_0^2 = \frac{c_0}{m} \qquad \omega_\alpha^2 = \frac{c_\alpha}{m} \ \omega_e^2 = \frac{c_e}{m} \qquad h_0 = \frac{F_0}{m} \tag{63}$$

and the previous systems of differential equations of fractional order I rewrite in the form:

* First

$$\ddot{x} + \omega_\alpha^2 \mathrm{D}_t^\alpha\left[x(t) - x_{u,2}(t)\right] = h_0 sin(\Omega t + \varphi_0) \tag{64}$$

$$-\omega_0^2\left(x_{u,2}(t) - x_{u,1}(t)\right) + \omega_e^2 x_{u,1}(t) = 0 \tag{65}$$

$$-\omega_\alpha^2 \mathrm{D}_t^\alpha\left[x(t) - x_{u,2}(t)\right] + \omega_0^2\left(x_{u,2}(t) - x_{u,1}(t)\right) = 0 \tag{66}$$

* Second

$$\ddot{x} + \omega_e^2\left(x(t) - x_{u,1}(t)\right) == h_0 sin(\Omega t + \varphi_0) \tag{67}$$

$$-\omega_e^2\left(x(t) - x_{u,1}(t)\right) - \omega_0^2\left(x_{u,2}(t) - x_{u,1}(t)\right) = 0 \tag{68}$$

$$-\omega_\alpha^2 \mathrm{D}_t^\alpha\left[x_{u,2}(t)\right] + \omega_o^2\left(x_{u,2}(t) - x_{u,1}(t)\right) = 0 \tag{69}$$

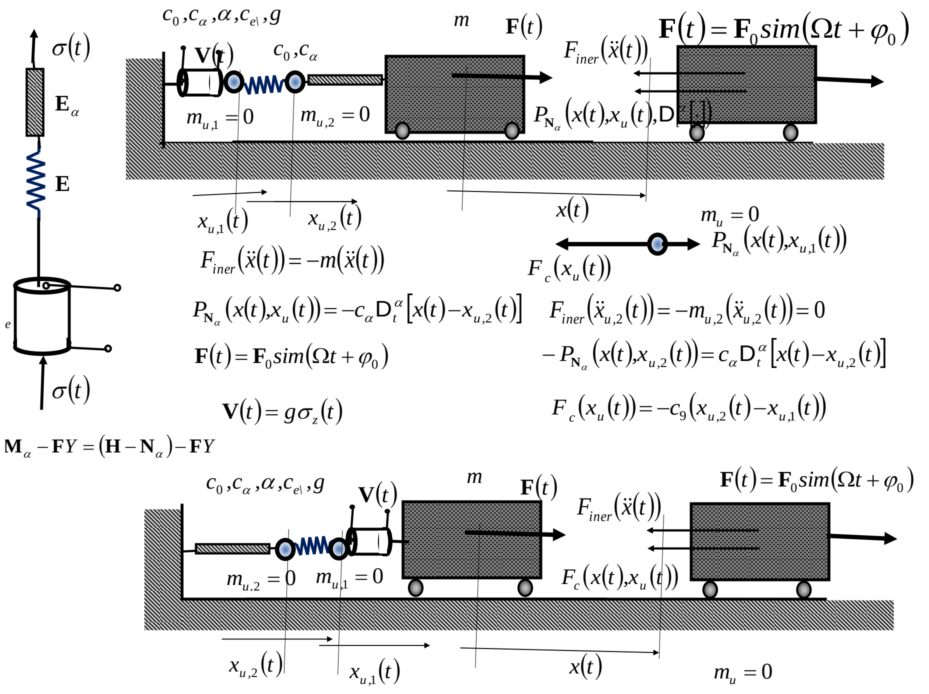


Fig. 13 Two fractional-type rheological Maxwell-Faraday discrete dynamical systems with piezoelectric properties (one rheological dynamical system model upper part and another rheological dynamical system model down part) and the structure of a standard easy basic rheological complex Maxwell-Faraday's bond model (left)

Now, let's apply the Laplace transformation to both previous systems of fractional type differential equations, so:

First, the system

$$\mathrm{L}\{x(t)\}\left\langle p^2+\omega_\alpha^2 p^\alpha\right\rangle-\omega_\alpha^2 p^\alpha \mathrm{L}\{x_{u,2}(t)\}=\frac{h_0\Omega}{p^2+\Omega^2}+\left(px_0+\dot{x}_0\right) \tag{70}$$

$$-\omega_0^2\mathrm{L}\{x_{u,2}(t)\}+\left(\omega_0^2+\omega_e^2\right)\mathrm{L}\{x_{u,1}(t)\}=0 \tag{71}$$

$$\mathrm{L}\{x_{u,2}(t)\}\left\langle \omega_\alpha^2 p^\alpha+\omega_0^2\right\rangle-\omega_0^2\mathrm{L}\{x_{u,1}(t)\}=\omega_\alpha^2 p^\alpha \mathrm{L}\{x(t)\} \tag{72}$$

This system of non-homogeneous algebraic equations is equal to the unknown Laplace transforms $\mathrm{L}\{x_{u,1}(t)\}$, $\mathrm{L}\{x(t)\}$ and $\mathrm{L}\{x_{u,2}(t)\}$, of unknown independent generalized coordinates $x(t)$, $x_{u,1}(t)$ and $x_{u,2}(t)$, and we can solve in two ways. One is by Cramer's rule and by the determinant of systems and modified determinants. The second one was obtained by eliminating the unknowns from the other two equations.

Now, let's go over the determinants of the system and Cramer's law. The determinant of the system is:

$$\Delta(p)=\begin{vmatrix} \left(p^2+\omega_\alpha^2 p^\alpha\right) & 0 & -\omega_\alpha^2 p^\alpha \\ 0 & \left(\omega_0^2+\omega_e^2\right) & -\omega_0^2 \\ -\omega_\alpha^2 p^\alpha & -\omega_0^2 & \left(\omega_\alpha^2 p^\alpha+\omega_0^2\right) \end{vmatrix}$$

$$\Delta(p)=\left(p^2+\omega_\alpha^2 p^\alpha\right)\left\langle\left(\omega_0^2+\omega_e^2\right)\left(\omega_\alpha^2 p^\alpha+\omega_0^2\right)-\omega_0^4\right\rangle-\omega_\alpha^4 p^{2\alpha}\left(\omega_0^2+\omega_e^2\right) \tag{73}$$

while the other determinants are obtained by replacing the corresponding column in the system determinant with a column of free members from the system algebra equations (70)-(72), so it follows:

$$\Delta_0(p)=\left\langle\frac{h_0\Omega}{p^2+\Omega^2}+\left(px_0+\dot{x}_0\right)\right\rangle\begin{vmatrix} 1 & 0 & -\omega_\alpha^2 p^\alpha \\ 0 & \left(\omega_0^2+\omega_e^2\right) & -\omega_0^2 \\ 0 & -\omega_0^2 & \left(\omega_\alpha^2 p^\alpha+\omega_0^2\right) \end{vmatrix}$$

$$\Delta_0(p)=\left\langle\frac{h_0\Omega}{p^2+\Omega^2}+\left(px_0+\dot{x}_0\right)\right\rangle\left\langle\left(\omega_0^2+\omega_e^2\right)\left(\omega_\alpha^2 p^\alpha+\omega_0^2\right)-\omega_0^4\right\rangle \tag{74}$$

$$\Delta_1(p)=\left\langle\frac{h_0\Omega}{p^2+\Omega^2}+\left(px_0+\dot{x}_0\right)\right\rangle\begin{vmatrix} \left(p^2+\omega_\alpha^2 p^\alpha\right) & 1 & -\omega_\alpha^2 p^\alpha \\ 0 & 0 & -\omega_0^2 \\ -\omega_\alpha^2 p^\alpha & 0 & \left(\omega_\alpha^2 p^\alpha+\omega_0^2\right) \end{vmatrix}$$

$$\Delta_1(p)=\omega_0^2\omega_\alpha^2 p^\alpha\left\langle\frac{h_0\Omega}{p^2+\Omega^2}+\left(px_0+\dot{x}_0\right)\right\rangle \tag{75}$$

$$\Delta_2(p)=\left\langle\frac{h_0\Omega}{p^2+\Omega^2}+\left(px_0+\dot{x}_0\right)\right\rangle\begin{vmatrix} \left(p^2+\omega_\alpha^2 p^\alpha\right) & 0 & 1 \\ 0 & \left(\omega_0^2+\omega_e^2\right) & 0 \\ -\omega_\alpha^2 p^\alpha & -\omega_0^2 & 0 \end{vmatrix}$$

$$\Delta_2(p)=\omega_\alpha^2 p^\alpha\left(\omega_0^2+\omega_e^2\right)\left\langle\frac{h_0\Omega}{p^2+\Omega^2}+\left(px_0+\dot{x}_0\right)\right\rangle \tag{76}$$

From there, the Laplace transforms $\mathrm{L}\{x(t)\}$, $\mathrm{L}\{x_{u,1}(t)\}$ and $\mathrm{L}\{x_{u,2}(t)\}$, of the unknown independent generalized coordinates $x(t)$, $x_{u,1}(t)$ and $x_{u,2}(t)$, and of the circle are unknown;

$$\mathrm{L}\{x(t)\}=\frac{\Delta_0(p)}{\Delta(p)}=\frac{\left\langle\left(\omega_0^2+\omega_e^2\right)\left(\omega_\alpha^2 p^\alpha+\omega_0^2\right)-\omega_0^4\right\rangle}{\Delta(p)}\left\langle\frac{h_0\Omega}{p^2+\Omega^2}+\left(px_0+\dot{x}_0\right)\right\rangle \tag{77}$$

$$\mathrm{L}\{x_{u,1}(t)\}=\frac{\Delta_1(p)}{\Delta(p)}=\frac{\omega_0^2\omega_\alpha^2 p^\alpha}{\Delta(p)}\left\langle\frac{h_0\Omega}{p^2+\Omega^2}+\left(px_0+\dot{x}_0\right)\right\rangle \tag{78}$$

$$\mathsf{L}\left\{x_{u,2}(t)\right\}=\frac{\Delta_2(p)}{\Delta(p)}=\frac{\omega_\alpha^2 p^\alpha\left(\omega_0^2+\omega_e^2\right)}{\Delta(p)}\left\langle\frac{h_0\Omega}{p^2+\Omega^2}+\left(px_0+\dot{x}_0\right)\right\rangle \tag{79}$$

Now it is necessary to determine the inverse Laplace transforms: $\mathsf{L}^{-1}\mathsf{L}\left\{x(t)\right\}$ the external independent generalized coordinates $\{x(t)\}$, as well as the inverse Laplace transforms $\mathsf{L}^{-1}\mathsf{L}\left\{x_{u,1}(t)\right\}$ and $\mathsf{L}^{-1}\mathsf{L}\left\{x_{u,2}(t)\right\}$, of the internal independent generalized coordinates $x_{u,1}(t)$ and $x_{u,2}(t)$, and move to the time domain, but we will not deal with that task.

The axial dilatation of the series-connected basic Faraday ideal elastic and piezoelectric element in the structure of the standard light Maxwell-Faraday's model, fractional type, with the piezoelectric property (see the model from the upper part of Fig. 13) is:

$$\varepsilon_z(t)=\frac{x_{u,1}(t)}{\ell_0}$$

$$\varepsilon_z(t)=\frac{x_{u,1}(t)}{\ell_0}=\frac{1}{\ell_0}\mathsf{L}^{-1}\mathsf{L}\left\{x_{u,1}(t)\right\}=\mathsf{L}^{-1}\left\{\frac{\omega_\alpha^2 p^\alpha}{\left(\omega_0^2+\omega_e^2\right)\left(\omega_\alpha^2 p^\alpha+\omega_0^2\right)-\omega_0^4}\right\}*\mathsf{L}^{-1}\mathsf{L}\left\{x(t)\right\} \tag{80}$$

The normal stress in the series-connected basic Faraday ideal elastic and piezoelectric element in the structure of the standard light Maxwell-Faraday's model, fractional type, with piezoelectric property (see the model from the upper part of Figure 13) is:

$$\sigma_z(t)=\mathbf{E_e}\varepsilon_z(t)=\mathbf{E_e}\frac{x_{u,1}(t)}{\ell_0}=\frac{\mathbf{E_e}}{\ell_0}\mathsf{L}^{-1}\mathsf{L}\left\{x_{u,1}(t)\right\}=\mathsf{L}^{-1}\left\{\frac{\omega_\alpha^2 p^\alpha}{\left(\omega_0^2+\omega_e^2\right)\left(\omega_\alpha^2 p^\alpha+\omega_0^2\right)-\omega_0^4}\right\}*\mathsf{L}^{-1}\mathsf{L}\left\{x(t)\right\} \tag{81}$$

The electric voltage of the electric polarization in the regularly connected basic Faraday's ideal elastic and piezoelectric element in the structure of the standard light Maxwell-Faraday's model, fractional type, with piezoelectric property (see the model from the upper part of Figure 16) is:

$$\mathbf{V}_z(t)=-g\sigma_z(t)=\frac{g\mathbf{E_e}}{\ell_0}\mathsf{L}^{-1}\mathsf{L}\left\{x_{u,1}(t)\right\}=\mathsf{L}^{-1}\left\{\frac{\omega_\alpha^2 p^\alpha}{\left(\omega_0^2+\omega_e^2\right)\left(\omega_\alpha^2 p^\alpha+\omega_0^2\right)-\omega_0^4}\right\}*\mathsf{L}^{-1}\mathsf{L}\left\{x(t)\right\} \tag{82}$$

While the dielectric displacement due to electric polarization in the series-connected basic Faraday ideal elastic and piezoelectric element in the structure of the standard light Maxwell-Faraday's model, fractional type, with piezoelectric property (see the model from the upper part of Figure 13) is:

$$D_z(t)=b\sigma_z(t)=\frac{b\mathbf{E_e}}{\ell_0}\mathsf{L}^{-1}\mathsf{L}\left\{x_{u,1}(t)\right\}=\mathsf{L}^{-1}\left\{\frac{\omega_\alpha^2 p^\alpha}{\left(\omega_0^2+\omega_e^2\right)\left(\omega_\alpha^2 p^\alpha+\omega_0^2\right)-\omega_0^4}\right\}*\mathsf{L}^{-1}\mathsf{L}\left\{x(t)\right\} \tag{83}$$

and

$$D_z(t)=e\varepsilon_z(t)=\frac{e}{\ell_0}\mathsf{L}^{-1}\mathsf{L}\left\{x_{u,1}(t)\right\}=\mathsf{L}^{-1}\left\{\frac{\omega_\alpha^2 p^\alpha}{\left(\omega_0^2+\omega_e^2\right)\left(\omega_\alpha^2 p^\alpha+\omega_0^2\right)-\omega_0^4}\right\}*\mathsf{L}^{-1}\mathsf{L}\left\{x(t)\right\}$$

For the opposite order of binding elements of the standard light rheological Maxwell-Faraday model, fractional type with piezo-electric roperty for the material point mass m, the model of the rheological Maxwell-Faraday dynamic system shown in the lower part of Fig. 13, by applying the Laplace transformation to the system of inhomogeneous differential fractional order equation (67)-(69), which describes the dynamics of that rheological dynamic system, fractional type with piezoelectric property, gives a system of inhomogeneous algebraic equations in the form:

$$\mathrm{L}\{x(t)\}\left(p^2+\omega_e^2\right)-\omega_e^2\mathrm{L}\{x_{u,1}(t)\}=\frac{h_0\Omega}{p^2+\Omega^2}+\left(px_0+\dot{x}_0\right) \tag{84}$$

$$\left(\omega_0^2+\omega_e^2\right)\mathrm{L}\{x_{u,1}(t)\}-\omega_0^2\mathrm{L}\{x_{u,2}(t)\}=\omega_e^2\mathrm{L}\{x(t)\} \tag{85}$$

$$\left(\omega_o^2-\omega_\alpha^2 p^\alpha\right)\mathrm{L}\{x_{u,2}(t)\}-\omega_o^2\mathrm{L}\{x_{u,1}(t)\}=0 \tag{86}$$

From the previous system (67)-(69) of three differential equations, to which we applied the Laplace transformation and obtained an inhomogeneous system of algebraic equations (84)-(86) by the unknown Laplace transformations of the unknown independent generalized coordinates: the Laplace transformation $\mathrm{L}\{x(t)\}$ of the external independent generalized coordinate $\{x(t)\}$, as well as the Laplace transformations $\mathrm{L}\{x_{u,1}(t)\}$ and $\mathrm{L}\{x_{u,2}(t)\}$, internal independent generalized coordinates $x_{u,1}(t)$ and $x_{u,2}(t)$. By solving that system of algebraic inhomogeneous equations using Cramer's rule for solutions, we obtain all three Laplace transforms: the Laplace transform $\mathrm{L}\{x(t)\}$ of the external independent generalized coordinate $\{x(t)\}$, as well as the Laplace transforms $\mathrm{L}\{x_{u,1}(t)\}$ and $\mathrm{L}\{x_{u,2}(t)\}$, the internal independent generalized coordinates $x_{u,1}(t)$ and $x_{u,2}(t)$, the dynamics of the rheological Maxwell-Faraday- of this dynamic system of the fractional type, shown in the lower part of the figure 16, with the piezoelectric property, the following form:

$$\begin{aligned}\mathrm{L}\{x(t)\}=&\frac{\left(px_0+\dot{x}_0\right)}{\left\langle\left(p^2+\omega_e^2\right)-\dfrac{\omega_e^4\left(\omega_o^2-\omega_\alpha^2p^\alpha\right)}{\left\langle\left(\omega_0^2+\omega_e^2\right)\left(\omega_o^2-\omega_\alpha^2p^\alpha\right)-\omega_o^4\right\rangle}\right\rangle}+\\&+\frac{1}{\left\langle\left(p^2+\omega_e^2\right)-\dfrac{\omega_e^4\left(\omega_o^2-\omega_\alpha^2p^\alpha\right)}{\left\langle\left(\omega_0^2+\omega_e^2\right)\left(\omega_o^2-\omega_\alpha^2p^\alpha\right)-\omega_o^4\right\rangle}\right\rangle}\frac{h_0\Omega}{p^2+\Omega^2}\end{aligned} \tag{87}$$

$$\mathrm{L}\{x_{u,1}(t)\}=\frac{\omega_e^2\left(\omega_o^2-\omega_\alpha^2p^\alpha\right)}{\left\langle\left(\omega_0^2+\omega_e^2\right)\left(\omega_o^2-\omega_\alpha^2p^\alpha\right)-\omega_o^4\right\rangle}\mathrm{L}\{x(t)\} \tag{88}$$

$$\mathrm{L}\{x_{u,2}(t)\}=\frac{\omega_o^2\omega_e^2}{\left\langle\left(\omega_0^2+\omega_e^2\right)\left(\omega_o^2-\omega_\alpha^2p^\alpha\right)-\omega_o^4\right\rangle}\mathrm{L}\{x(t)\} \tag{89}$$

Now it is necessary to determine the inverse Laplace transformations: $\mathrm{L}^{-1}\mathrm{L}\{x(t)\}$ external independent generalized coordinate $\{x(t)\}$, as well as inverse Laplace transformations $\mathsf{L}^{-1}\mathsf{L}\{x_{u,1}(t)\}$ and $\mathsf{L}^{-1}\mathsf{L}\{x_{u,2}(t)\}$, internal independent generalized coordinates $x_{u,1}(t)$ and $x_{u,2}(t)$, and move to the time domain, but in this work we will not on that, because it is a new not an easy math task at all.

The axial expansion of the serially connected basic Faraday ideally elastic and piezoelectric element in the structure of the standard light Maxwell-Faraday's model, fractional type, with piezoelectric property (see the model from the lower part of Figure 16) is:

$$\varepsilon_z(t)=\frac{x(t)-x_{u,1}(t)}{\ell_0}=\frac{1}{\ell_0}\mathsf{L}^{-1}\mathsf{L}\{x(t)-x_{u,1}(t)\} \tag{90}$$

The normal stress in the regularly connected basic Faraday's ideally elastic and piezoelectric element in the structure of the standard light Maxwell-Faraday's model, fractional type, with piezoelectric property (see the model from the lower part of Figure 16) is:

$$\sigma_z(t)=\mathbf{E_e}\varepsilon_z(t)=\mathbf{E_e}\frac{x(t)-x_{u,1}(t)}{\ell_0}=\frac{\mathbf{E_e}}{\ell_0}\mathsf{L}^{-1}\mathsf{L}\{x(t)-x_{u,1}(t)\} \tag{91}$$

The electric polarization voltage in the regularly connected fundamental Faraday ideally elastic and piezoelectric element in the structure of the standard light Maxwell-Faraday's model, fractional type, with piezoelectric property (see the model from the lower part of Figure 13) is:

$$\mathbf{V}_z(t)=-g\sigma_z(t)=g\mathbf{E_e}\frac{x(t)-x_{u,1}(t)}{\ell_0}=g\frac{\mathbf{E_e}}{\ell_0}\mathsf{L}^{-1}\mathsf{L}\{x(t)-x_{u,1}(t)\} \tag{92}$$

While the dielectric displacement due to electric polarization in the regularly connected basic Faraday ideally elastic and piezoelectric element in the structure of the standard light Maxwell-Faraday's model, fractional type, with piezoelectric property (see the model from the lower part of Figure 13) is:

$$D_z(t)=b\sigma_z(t)=b\mathbf{E_e}\frac{x(t)-x_{u,1}(t)}{\ell_0}=b\frac{\mathbf{E_e}}{\ell_0}\mathsf{L}^{-1}\mathsf{L}\{x(t)-x_{u,1}(t)\} \tag{93}$$

and

$$D_z(t)=e\varepsilon_z(t)=e\frac{x(t)-x_{u,1}(t)}{\ell_0}=\frac{e}{\ell_0}\mathsf{L}^{-1}\mathsf{L}\{x(t)-x_{u,1}(t)\}$$

By comparing the obtained expressions for both natural and forced motions of a material point (rigid body), we see that the binding order of this structure of the standard light rheological Maxwell-Faraday model, fractional type, has a great influence on the resulting motion of the rheological Maxwell-Faraday dynamic system. In both cases, the sequence of binding the structure to the resulting movement is the creep of the material point, with the property of stress relaxation, and the dominant influence is on the piezo-electric polarization, as well as on the electric voltage of the electric field, as a consequence of dilation and stress.

3.2.1. The graphical presentation of some characteristic properties of dynamics of the rheological Maxwell-Faraday's dynamical systems fractional type with piezoectrical property

In this subsection, we also investigate several properties of the Laplace transforms of the eigenmodes and forced modes of the dynamics of a rheological Maxwell–Faraday dynamical system of crawler (creep) type, of fractional order and with piezoelectric properties. These results are illustrated by spatial surfaces represented in coordinate systems whose axes correspond to: the Laplace transform of the elongation, the fractional-order differentiation exponent αin the interval $0 < \alpha < 1$, and the Laplace transform parameter p.

Accordingly, we construct series of sets of Laplace transforms corresponding to the eigenmodes and forced modes of the crawler–creep dynamics of the fractional-order rheological Maxwell–Faraday crawler–creep system with piezoelectric properties.

The set of Laplace transforms $\mathsf{L}\left\{x_{free,likecos}\right\}$ and $\mathsf{L}\left\{x_{free,likesin}\right\}$ of the eigenmodes associated with the independent generalized coordinate of the external degree of freedom—in the unique dynamics of the fractional-order rheological Maxwell–Faraday crawler–creep dynamical system with piezoelectric properties, depicted in the upper part of Fig. 13—are given by:

$$\mathsf{L}\left\{x_{free,likesin}(t)\right\}=\frac{1}{\left\langle p^{2}+\omega_{\alpha}^{2}p^{\alpha}-\frac{\omega_{\alpha}^{4}p^{2\alpha}}{\left(\omega_{0}^{2}+\omega_{e}^{2}\right)\left(\omega_{\alpha}^{2}p^{\alpha}+\omega_{0}^{2}\right)-\omega_{0}^{4}}\right\rangle} \tag{94}$$

$$\mathsf{L}\left\{x_{free,likecos}(t)\right\}=\frac{p}{\left\langle p^{2}+\omega_{\alpha}^{2}p^{\alpha}-\frac{\omega_{\alpha}^{4}p^{2\alpha}}{\left(\omega_{0}^{2}+\omega_{e}^{2}\right)\left(\omega_{\alpha}^{2}p^{\alpha}+\omega_{0}^{2}\right)-\omega_{0}^{4}}\right\rangle} \tag{95}$$

The set of Laplace transforms $\mathsf{L}\left\{x_{Forced,sin}(t)\right\}$ and $\mathsf{L}\left\{x_{Forced,cos}(t)\right\}$ of the forced modes of independent generalized coordinate of external degree of freedom of dynamics of the rheological Maxwell--Faraday rheological crawler-creep dynamic system, fractional type with piezoelectric property, presented in upper part of Fig. 13, is:

$$\mathsf{L}\left\{x_{Forced,sin}(t)\right\}=\frac{1}{\left\langle p^{2}+\omega_{\alpha}^{2}p^{\alpha}-\frac{\omega_{\alpha}^{4}p^{2\alpha}}{\left(\omega_{0}^{2}+\omega_{e}^{2}\right)\left(\omega_{\alpha}^{2}p^{\alpha}+\omega_{0}^{2}\right)-\omega_{0}^{4}}\right\rangle}\frac{\Omega}{p^{2}+\Omega^{2}} \tag{96}$$

$$\mathsf{L}\left\{x_{Forced,cos}(t)\right\}=\frac{1}{\left\langle p^{2}+\omega_{\alpha}^{2}p^{\alpha}-\frac{\omega_{\alpha}^{4}p^{2\alpha}}{\left(\omega_{0}^{2}+\omega_{e}^{2}\right)\left(\omega_{\alpha}^{2}p^{\alpha}+\omega_{0}^{2}\right)-\omega_{0}^{4}}\right\rangle}\frac{p}{p^{2}+\Omega^{2}} \tag{97}$$

The set of Laplace transforms $\mathsf{L}\left\{x_{u,1,free,likesin}(t)\right\}$ and $\mathsf{L}\left\{x_{u,1,free,likecos}(t)\right\}$ of the eigenmodes of independent generalized coordinate of first internal degree of freedom, of dynamics of the rheological Maxwell--Faraday rheological crawler-creep dynamic system, fractional type with piezoelectric property, presented in upper part of Fig. 13, is:

$$\mathrm{L}\left\{x_{u,1,free,likesin}(t)\right\}=\frac{\omega_\alpha^2 p^\alpha}{\left(\omega_0^2+\omega_e^2\right)\left(\omega_\alpha^2 p^\alpha+\omega_0^2\right)-\omega_0^4}\frac{1}{\left\langle p^2+\omega_\alpha^2 p^\alpha-\frac{\omega_\alpha^4 p^{2\alpha}}{\left(\omega_0^2+\omega_e^2\right)\left(\omega_\alpha^2 p^\alpha+\omega_0^2\right)-\omega_0^4}\right\rangle} \tag{98}$$

$$\mathrm{L}\left\{x_{u,1,free,likecos}(t)\right\}=\frac{\omega_\alpha^2 p^\alpha}{\left(\omega_0^2+\omega_e^2\right)\left(\omega_\alpha^2 p^\alpha+\omega_0^2\right)-\omega_0^4}\frac{p}{\left\langle p^2+\omega_\alpha^2 p^\alpha-\frac{\omega_\alpha^4 p^{2\alpha}}{\left(\omega_0^2+\omega_e^2\right)\left(\omega_\alpha^2 p^\alpha+\omega_0^2\right)-\omega_0^4}\right\rangle} \tag{99}$$

The set of Laplace transforms $\mathrm{L}\left\{x_{u,1,Forced,sin}(t)\right\}$ and $\mathrm{L}\left\{x_{u,1,Forced,cos}(t)\right\}$ of the forced modes of independent generalized coordinate of first internal degree of freedom, of dynamics of the rheological Maxwell--Faraday rheological crawler-creep dynamic system, fractional type with piezoelectric property, presented in upper part of Fig. 13, is:

$$\mathrm{L}\left\{x_{u,1,Forced,sin}(t)\right\}=\frac{\omega_\alpha^2 p^\alpha}{\left(\omega_0^2+\omega_e^2\right)\left(\omega_\alpha^2 p^\alpha+\omega_0^2\right)-\omega_0^4}\frac{1}{\left\langle p^2+\omega_\alpha^2 p^\alpha-\frac{\omega_\alpha^4 p^{2\alpha}}{\left(\omega_0^2+\omega_e^2\right)\left(\omega_\alpha^2 p^\alpha+\omega_0^2\right)-\omega_0^4}\right\rangle}\frac{\Omega}{p^2+\Omega^2} \tag{100}$$

$$\mathrm{L}\left\{x_{u,1,Forced,cos}(t)\right\}=\frac{\omega_\alpha^2 p^\alpha}{\left(\omega_0^2+\omega_e^2\right)\left(\omega_\alpha^2 p^\alpha+\omega_0^2\right)-\omega_0^4}\frac{1}{\left\langle p^2+\omega_\alpha^2 p^\alpha-\frac{\omega_\alpha^4 p^{2\alpha}}{\left(\omega_0^2+\omega_e^2\right)\left(\omega_\alpha^2 p^\alpha+\omega_0^2\right)-\omega_0^4}\right\rangle}\frac{p}{p^2+\Omega^2} \tag*{101)}$$

The set of Laplace transforms $\mathrm{L}\left\{x_{u,2,free,likesin}(t)\right\}$ and $\mathrm{L}\left\{x_{u,2,free,likecos}(t)\right\}$ of the eigenmodes of independent generalized coordinate of second internal degree of freedom, of dynamics of the rheological Maxwell--Faraday rheological crawler-creep dynamic system, fractional type with piezoelectric property, presented in upper part of Fig. 13, is:

$$\mathrm{L}\left\{x_{u,2,free,likesin}(t)\right\}=\frac{\left(\omega_0^2+\omega_e^2\right)\omega_\alpha^2 p^\alpha}{\omega_0^2\left\langle\left(\omega_0^2+\omega_e^2\right)\left(\omega_\alpha^2 p^\alpha+\omega_0^2\right)-\omega_0^4\right\rangle}\frac{1}{\left\langle p^2+\omega_\alpha^2 p^\alpha-\frac{\omega_\alpha^4 p^{2\alpha}}{\left(\omega_0^2+\omega_e^2\right)\left(\omega_\alpha^2 p^\alpha+\omega_0^2\right)-\omega_0^4}\right\rangle} \tag{102}$$

$$\mathrm{L}\left\{x_{u,2,free,likecos}(t)\right\}=\frac{\left(\omega_0^2+\omega_e^2\right)\omega_\alpha^2 p^\alpha}{\omega_0^2\left\langle\left(\omega_0^2+\omega_e^2\right)\left(\omega_\alpha^2 p^\alpha+\omega_0^2\right)-\omega_0^4\right\rangle}\frac{p}{\left\langle p^2+\omega_\alpha^2 p^\alpha-\frac{\omega_\alpha^4 p^{2\alpha}}{\left(\omega_0^2+\omega_e^2\right)\left(\omega_\alpha^2 p^\alpha+\omega_0^2\right)-\omega_0^4}\right\rangle} \tag{103}$$

The set of Laplace transforms $\mathrm{L}\left\{x_{u,2,Forced,sin}(t)\right\}$ and $\mathrm{L}\left\{x_{u,2,Forced,cos}(t)\right\}$ of the forced modes of independent generalized coordinate of second internal degree of freedom, of dynamics of the rheological Maxwell--Faraday rheological crawler-creep dynamic system, fractional type with piezoelectric property, presented in upper part of Fig. 13, is:

$$\mathrm{L}\left\{x_{u,2,Forced,sin}(t)\right\}=\frac{\left(\omega_0^2+\omega_e^2\right)\omega_\alpha^2 p^\alpha}{\omega_0^2\left\langle\left(\omega_0^2+\omega_e^2\right)\left(\omega_\alpha^2 p^\alpha+\omega_0^2\right)-\omega_0^4\right\rangle}\frac{1}{\left\langle p^2+\omega_\alpha^2 p^\alpha-\frac{\omega_\alpha^4 p^{2\alpha}}{\left(\omega_0^2+\omega_e^2\right)\left(\omega_\alpha^2 p^\alpha+\omega_0^2\right)-\omega_0^4}\right\rangle}\frac{\Omega}{p^2+\Omega^2} \tag{104}$$

$$\mathrm{L}\left\{x_{u,2,Forced,cos}(t)\right\}=\frac{\left(\omega_0^2+\omega_e^2\right)\omega_\alpha^2 p^\alpha}{\omega_0^2\left\langle\left(\omega_0^2+\omega_e^2\right)\left(\omega_\alpha^2 p^\alpha+\omega_0^2\right)-\omega_0^4\right\rangle}\frac{1}{\left\langle p^2+\omega_\alpha^2 p^\alpha-\frac{\omega_\alpha^4 p^{2\alpha}}{\left(\omega_0^2+\omega_e^2\right)\left(\omega_\alpha^2 p^\alpha+\omega_0^2\right)-\omega_0^4}\right\rangle}\frac{p}{p^2+\Omega^2} \tag{105}$$

The set of Laplace transforms $\mathrm{L}\left\{x_{free,likecos}\right\}$ and $\mathrm{L}\left\{x_{free,likesin}\right\}$ of the eigenmodes of independent generalized coordinate of external degree of freedom, of unique dynamics of the rheological Maxwell--Faraday rheological crawler-creep dynamic system, fractional type with piezoelectric property, presented in dawn part of Fig. 13, is:

$$\mathrm{L}\left\{x_{free,likesin}(t)\right\}=\frac{1}{\left\langle\left(p^2+\omega_e^2\right)-\dfrac{\omega_e^4\left(\omega_o^2-\omega_\alpha^2 p^\alpha\right)}{\left\langle\left(\omega_0^2+\omega_e^2\right)\left(\omega_o^2-\omega_\alpha^2 p^\alpha\right)-\omega_o^4\right\rangle}\right\rangle} \tag{106}$$

$$\mathrm{L}\left\{x_{free,likecos}(t)\right\}=\frac{p}{\left\langle\left(p^2+\omega_e^2\right)-\dfrac{\omega_e^4\left(\omega_o^2-\omega_\alpha^2 p^\alpha\right)}{\left\langle\left(\omega_0^2+\omega_e^2\right)\left(\omega_o^2-\omega_\alpha^2 p^\alpha\right)-\omega_o^4\right\rangle}\right\rangle} \tag{107}$$

The set of Laplace transforms $\mathrm{L}\left\{x_{Forced,sin}(t)\right\}$ and $\mathrm{L}\left\{x_{Forced,cos}(t)\right\}$ of the forced modes of independent generalized coordinate of external degree of freedom of dynamics of the rheological Maxwell--Faraday rheological crawler-creep dynamic system, fractional type with piezoelectric property, presented in dawn part of Fig. 13, is:

$$\mathrm{L}\left\{x_{Forced,sin}(t)\right\}=\frac{1}{\left\langle\left(p^2+\omega_e^2\right)-\dfrac{\omega_e^4\left(\omega_o^2-\omega_\alpha^2 p^\alpha\right)}{\left\langle\left(\omega_0^2+\omega_e^2\right)\left(\omega_o^2-\omega_\alpha^2 p^\alpha\right)-\omega_o^4\right\rangle}\right\rangle}\frac{\Omega}{p^2+\Omega^2} \tag{108}$$

$$\mathrm{L}\left\{x_{Forced,cos}(t)\right\}=\frac{1}{\left\langle\left(p^2+\omega_e^2\right)-\dfrac{\omega_e^4\left(\omega_o^2-\omega_\alpha^2 p^\alpha\right)}{\left\langle\left(\omega_0^2+\omega_e^2\right)\left(\omega_o^2-\omega_\alpha^2 p^\alpha\right)-\omega_o^4\right\rangle}\right\rangle}\frac{p}{p^2+\Omega^2} \tag{109}$$

The set of Laplace transforms $\mathrm{L}\left\{x_{u,1,free,likesin}(t)\right\}$ and $\mathrm{L}\left\{x_{u,1,free,likecos}(t)\right\}$ of the eigenmodes of independent generalized coordinate of first internal degree of freedom, of dynamics of the rheological Maxwell--Faraday rheological crawler-creep dynamic system, fractional type with piezoelectric property, presented in dawn part of Fig. 13, is:

$$\mathrm{L}\left\{x_{u,1,free,likesin}(t)\right\}=\frac{\omega_e^2\left(\omega_o^2-\omega_\alpha^2 p^\alpha\right)}{\left\langle\left(\omega_0^2+\omega_e^2\right)\left(\omega_o^2-\omega_\alpha^2 p^\alpha\right)-\omega_o^4\right\rangle}\frac{1}{\left\langle\left(p^2+\omega_e^2\right)-\dfrac{\omega_e^4\left(\omega_o^2-\omega_\alpha^2 p^\alpha\right)}{\left\langle\left(\omega_0^2+\omega_e^2\right)\left(\omega_o^2-\omega_\alpha^2 p^\alpha\right)-\omega_o^4\right\rangle}\right\rangle} \tag{110}$$

$$\mathrm{L}\left\{x_{u,1,free,likecos}(t)\right\}=\frac{\omega_e^2\left(\omega_o^2-\omega_\alpha^2 p^\alpha\right)}{\left\langle\left(\omega_0^2+\omega_e^2\right)\left(\omega_o^2-\omega_\alpha^2 p^\alpha\right)-\omega_o^4\right\rangle}\frac{p}{\left\langle\left(p^2+\omega_e^2\right)-\dfrac{\omega_e^4\left(\omega_o^2-\omega_\alpha^2 p^\alpha\right)}{\left\langle\left(\omega_0^2+\omega_e^2\right)\left(\omega_o^2-\omega_\alpha^2 p^\alpha\right)-\omega_o^4\right\rangle}\right\rangle} \tag{111}$$

The set of Laplace transforms $\mathrm{L}\left\{x_{u,2,Forced,sin}(t)\right\}$ and $\mathrm{L}\left\{x_{u,2,Forced,cos}(t)\right\}$ of the forced modes of independent generalized coordinate of second internal degree of freedom, of dynamics of the rheological Maxwell--Faraday rheological crawler-creep dynamic system, fractional type with piezoelectric property, presented in dawn part of Fig. 13, is:

$$\mathrm{L}\left\{x_{u,2,Forced,sin}(t)\right\}=\frac{\omega_o^2\omega_e^2}{\left\langle\left(\omega_0^2+\omega_e^2\right)\left(\omega_o^2-\omega_\alpha^2 p^\alpha\right)-\omega_o^4\right\rangle}\frac{1}{\left\langle\left(p^2+\omega_e^2\right)-\dfrac{\omega_e^4\left(\omega_o^2-\omega_\alpha^2 p^\alpha\right)}{\left\langle\left(\omega_0^2+\omega_e^2\right)\left(\omega_o^2-\omega_\alpha^2 p^\alpha\right)-\omega_o^4\right\rangle}\right\rangle}\frac{\Omega}{p^2+\Omega^2} \tag{112}$$

$$\mathcal{L}\{x_{u,2,Forced,cos}(t)\}=\frac{\omega_o^2\omega_e^2}{\left\langle\left(\omega_0^2+\omega_e^2\right)\left(\omega_o^2-\omega_\alpha^2 p^\alpha\right)-\omega_o^4\right\rangle}\frac{1}{\left\langle\left(p^2+\omega_e^2\right)-\dfrac{\omega_e^4\left(\omega_o^2-\omega_\alpha^2 p^\alpha\right)}{\left\langle\left(\omega_0^2+\omega_e^2\right)\left(\omega_o^2-\omega_\alpha^2 p^\alpha\right)-\omega_o^4\right\rangle}\right\rangle}\frac{p}{p^2+\Omega^2} \tag{113}$$

The set of Laplace transforms $\mathcal{L}\{x_{u,2,free,likesin}(t)\}$ and $\mathcal{L}\{x_{u,2,free,likecos}(t)\}$ of the eigenmodes of independent generalized coordinate of second internal degree of freedom, of dynamics of the rheological Maxwell--Faraday rheological crawler-creep dynamic system, fractional type with piezoelectric property, presented in dawn part of Fig. 13, is:

$$\mathcal{L}\{x_{u,2,free,likesin}(t)\}=\frac{\omega_o^2\omega_e^2}{\left\langle\left(\omega_0^2+\omega_e^2\right)\left(\omega_o^2-\omega_\alpha^2 p^\alpha\right)-\omega_o^4\right\rangle}\frac{1}{\left\langle\left(p^2+\omega_e^2\right)-\dfrac{\omega_e^4\left(\omega_o^2-\omega_\alpha^2 p^\alpha\right)}{\left\langle\left(\omega_0^2+\omega_e^2\right)\left(\omega_o^2-\omega_\alpha^2 p^\alpha\right)-\omega_o^4\right\rangle}\right\rangle} \tag{114}$$

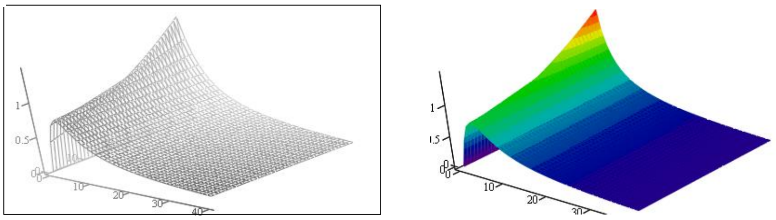
$$\mathcal{L}\{x_{u,2,free,likecos}(t)\}=\frac{\omega_o^2\omega_e^2}{\left\langle\left(\omega_0^2+\omega_e^2\right)\left(\omega_o^2-\omega_\alpha^2 p^\alpha\right)-\omega_o^4\right\rangle}\frac{p}{\left\langle\left(p^2+\omega_e^2\right)-\dfrac{\omega_e^4\left(\omega_o^2-\omega_\alpha^2 p^\alpha\right)}{\left\langle\left(\omega_0^2+\omega_e^2\right)\left(\omega_o^2-\omega_\alpha^2 p^\alpha\right)-\omega_o^4\right\rangle}\right\rangle} \tag{115}$$

The set of Laplace transforms $\mathcal{L}\{x_{u,2,Forced,sin}(t)\}$ and $\mathcal{L}\{x_{u,2,Forced,cos}(t)\}$ of the forced modes of independent generalized coordinate of second internal degree of freedom, of dynamics of the rheological Maxwell--Faraday rheological crawler-creep dynamic system, fractional type with piezoelectric property, presented in dawn part of Figure 13, is:

$$\mathcal{L}\{x_{u,2,Forced,sin}(t)\}=\frac{\omega_o^2\omega_e^2}{\left\langle\left(\omega_0^2+\omega_e^2\right)\left(\omega_o^2-\omega_\alpha^2 p^\alpha\right)-\omega_o^4\right\rangle}\frac{1}{\left\langle\left(p^2+\omega_e^2\right)-\dfrac{\omega_e^4\left(\omega_o^2-\omega_\alpha^2 p^\alpha\right)}{\left\langle\left(\omega_0^2+\omega_e^2\right)\left(\omega_o^2-\omega_\alpha^2 p^\alpha\right)-\omega_o^4\right\rangle}\right\rangle}\frac{\Omega}{p^2+\Omega^2} \tag{116}$$

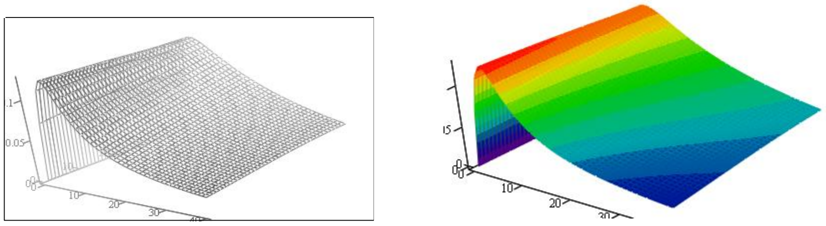
$$\mathcal{L}\{x_{u,2,Forced,cos}(t)\}=\frac{\omega_o^2\omega_e^2}{\left\langle\left(\omega_0^2+\omega_e^2\right)\left(\omega_o^2-\omega_\alpha^2 p^\alpha\right)-\omega_o^4\right\rangle}\frac{1}{\left\langle\left(p^2+\omega_e^2\right)-\dfrac{\omega_e^4\left(\omega_o^2-\omega_\alpha^2 p^\alpha\right)}{\left\langle\left(\omega_0^2+\omega_e^2\right)\left(\omega_o^2-\omega_\alpha^2 p^\alpha\right)-\omega_o^4\right\rangle}\right\rangle}\frac{p}{p^2+\Omega^2} \tag{117}$$

In the next few Figs 14, 15 and 16, we will show only a small selection of the characteristic surfaces of the Laplace transforms of the eigen and forced modes of creep dynamics of a rheological Maxwell-Faraday dynamic system of crawler-creeper type, fractional type with piezoelectric property, for external and internal creep degrees of freedom.



a M b M

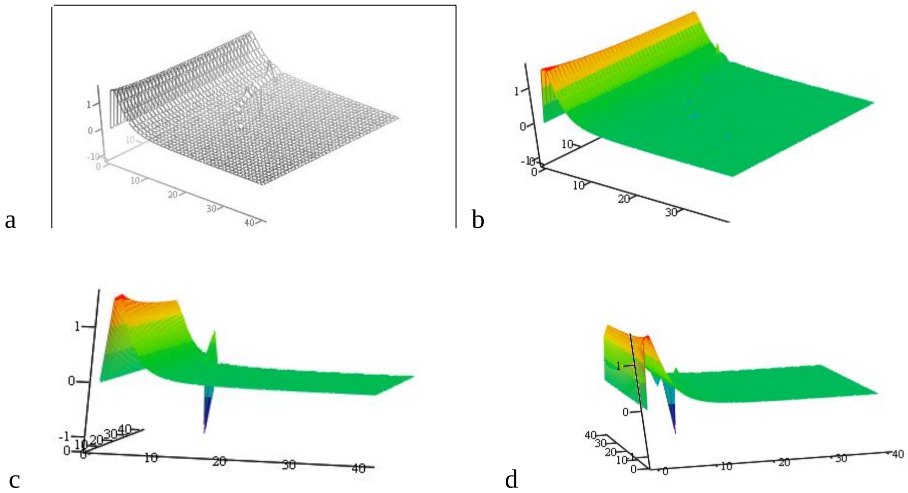
$\mathcal{L}\{x_{free,likesin}\}$



c M d M

$L\{x_{free,likeos}\}$

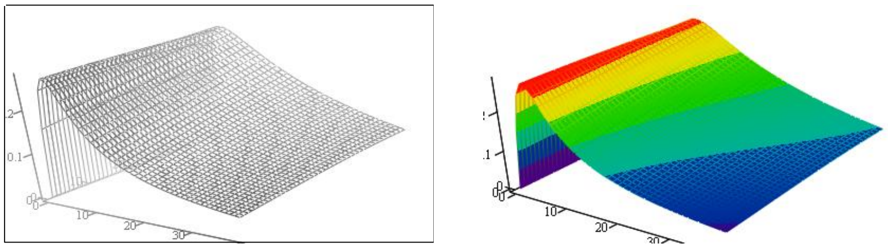
Fig. 14. Surfaces of Laplace transforms of eigenmodes, $L\{x_{free,likesin}\}$ and $L\{x_{free,likeos}\}$ fractional-type crawler, depending on the fractional-order differentiation exponent α in the interval $0 < \alpha < 1$, and the Laplace transforms parameter p, for the crawler-creep dynamics of the rheological Maxwell-Faraday's dynamic system of crawler-creeper type, fractional type with piezoelectric property, presented in upper part from Fig. 13, drawn using analytical expressions (94) and (95)



a b

c c d

Fig. 15. Surfaces of Laplace transforms of forced mode, $L\{x_{Forced,sin}(t)\}$ fractional-type crawler-creep motion, depending on the differentiation exponent α of the fractional order in the interval $0 < \alpha < 1$, and the parameter p of Laplace transforms, for forced crawler-creep motion dynamics of the rheological Maxwell's dynamic system, presented in upper part from Fig. 13, drawn using the analytical expression (96), and the cases of single-frequency periodic force action $F(t) = F_0 sin(\Omega t + \phi_0)$



a M b M

Fig. 16 Surfaces of Laplace transforms of forced mode, $\mathcal{L}\{x_{Forced,likwcos}\}$ fractional-type crawler-creep motion, depending on the differentiation exponent α of the fractional order in the interval $0 < \alpha < 1$, and the parameter p of Laplace transforms, for forced crawler-creep motion dynamics of the rheological Maxwell-Faraday's crawler-creep dynamic system, presented in dawn part from Fig. 13, drawn using the analytical expression (108), and the cases of single-frequency periodic force action $F(t) = F_0 sin(\Omega t + \phi_0)$

4. CONCLUSIONS

In concluding this paper, we summarize the main new results obtained in Subsections 2.1 and 2.2, where fractional-order differential constitutive relations were derived for two standard lightweight rheological binding models: the fractional Kelvin–Voigt–Faraday model with piezoelectric properties and the fractional Maxwell–Faraday model with piezoelectric properties. It was demonstrated that the Kelvin–Voigt–Faraday model exhibits the property of delayed (subsequent) elasticity, whereas the Maxwell–Faraday model possesses the property of fractional stress relaxation. A comparative illustration of these characteristic properties of the newly defined standard lightweight rheological models is presented in Fig. 3.

Based on the results of the dynamic analysis presented in Chapter 3—specifically, the rheological oscillator analyzed in Subsection 3.1 and the rheological crawler (creep system) analyzed in Subsection 3.2—we formulate the following general result.

Theorem (Difference between rheological oscillatory and creeping dynamical systems of fractional order with piezoelectric properties)

If at least one Newtonian fluid element of fractional type is connected in a regular serial manner within the structure—without being arranged in parallel or intra-parallel connection with any Hooke ideally elastic element or Faraday ideally elastic piezoelectric element—then the resulting rheological dynamical system exhibits creep (yielding or flow) dynamics and represents a complex creeping rheological model.

Conversely, if each Newtonian fluid element of fractional type is connected in parallel or intra-parallel configuration with a Hooke ideally elastic element or a Faraday ideally elastic piezoelectric element, then the corresponding rheological dynamical system exhibits elastoviscous damped oscillatory behavior.

Furthermore, each regular serial connection of a Newtonian fractional-type fluid element, a Hooke ideally elastic element, or a Faraday ideally elastic piezoelectric element introduces one internal degree of freedom in the rheological complex model, in addition to the external degrees of freedom of the rheological dynamical system containing standard lightweight rheological binding elements.

In the concluding part of the study, we also emphasize the analogies between mechanical rheological dynamical systems and their corresponding electrical dynamical counterparts. Based on these analogies, we introduce the following new terms:

-the rheological Kelvin–Voigt–Faraday electrical dynamical system, termed the *fractional rheological electric oscillator with piezoelectric properties* (Figs. 4 and 17), and

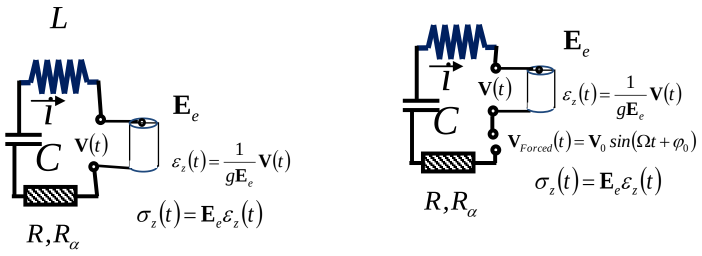
-the rheological Maxwell–Faraday electrical dynamical system, termed the *fractional rheological electric crawler with piezoelectric properties* (Figs. 13 and 18).

Clear mathematical analogies are established between the dynamics of the corresponding mechanical and electrical rheological systems of fractional order with piezoelectric coupling. In particular:

-the rheological Kelvin–Voigt–Faraday electrical oscillator (Fig. 17) is mathematically analogous to the rheological Kelvin–Voigt–Faraday mechanical oscillator (see Fig. 4 and the upper part of Fig. 13), and

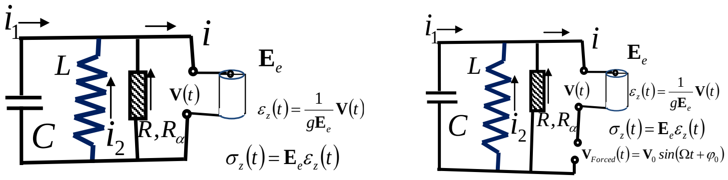
-the rheological Maxwell–Faraday electrical crawler (Fig. 21) is mathematically analogous to the rheological Maxwell–Faraday mechanical crawler (see Fig. 18 and the lower part of Fig. 13).

Finally, new generalized complex fractional-order rheological electrical circuits are introduced, incorporating generalized fractional resistor elements and Faraday piezoelectric elements, as illustrated in Fig. 17. These circuits provide a unified phenomenological framework for modeling coupled mechanical–electrical dynamical processes in fractional-order rheological systems.



a b

Fig. 17 Generalized rheologic basic complex Kelvin-Voigt- Faraday's, fractional type, dynamic electric system of elasto-viscous electric oscillator, with one degree of freedom, with property of polarization of Faraday's element with oscillatory dilatations a) for free electric oscillations and b) for forced electric oscillations. Independent generalized coordinate electric current i in electric circle; Property of relaxation of electric voltage when electric current is constant (part of electric circle without inductive coil).



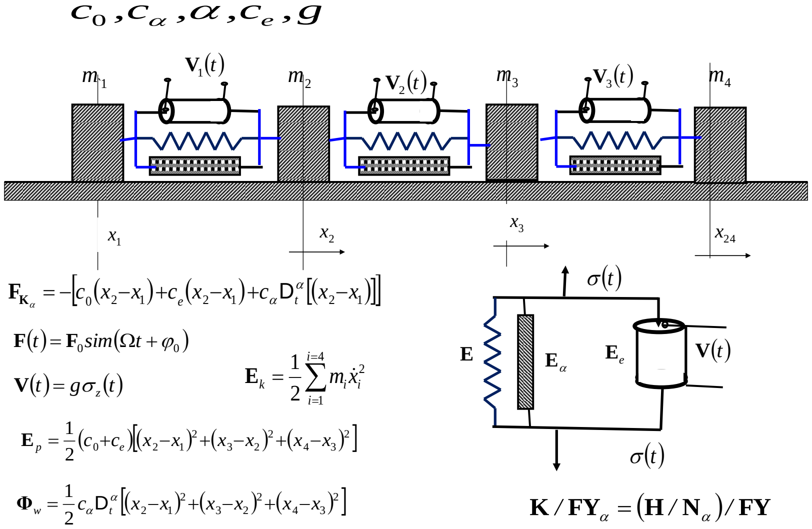
a b

Fig. 18. Generalized rheologic complex Maxwell-Faraday's, fractional type, dynamic system of viscoelastic creep motion model of elasto-viscous electric crawler, with one external degree of freedom, and two internal degrees of freedom, and with property of polarization of Faraday's element with creep dilatations a) for free creep motion in electric dynamical system and b) for forced electric creeping motion; Independent generalized coordinates electric currents, external coordinate electric current i and internal coordinates two electric currents i_1 and i_2 in electric circle; Property of the subsequent electric current when electric voltage is constant (part of electric circle without inductive coil).

These two electrical circuits may also be regarded as conceptual designs of piezoelectric ultrasonic exciters (transducers) intended for various technical applications (see Refs. [7, 9, 10]). If mechanical concentrators—such as rods with variable cross-section (see Refs. [11, 12])—are attached to the rheological Faraday element, thereby amplifying the amplitudes of longitudinal oscillations at the tip (see Fig. in Ref. [10]), a construction scheme is obtained for devices such as ultrasonic systems for homogenizing components of phases that do not readily mix, for example, metallic alloys.

Furthermore, when such a system is attached to a tub, it can be used as a device for cleaning and surface degradation of machine parts processed on a lathe using an oil jet, or as an ultrasonic sprayer if a fluid is passed through the internal channel of the concentrator (see Fig. in Ref. [10]).

Finally, about further research directions, we point to the investigation of the dynamics of fractional-order rheological dynamical systems with piezoelectric properties and a finite number of degrees of freedom, of either rheological oscillator or rheological crawler type. One such research direction is illustrated in Figs. 22 and 23, where rheological homogeneous chains are depicted: a rheological complex Kelvin–Voigt–Faraday homogeneous chain of oscillator type (Fig. 19) and a rheological complex Maxwell–Faraday homogeneous chain of crawler type (Fig. 20).



$$\mathbf{F}_{\mathbf{K}_\alpha} = -\left[c_0(x_2-x_1)+c_e(x_2-x_1)+c_\alpha \mathsf{D}_t^\alpha[(x_2-x_1)]\right]$$

$$\mathbf{F}(t)=\mathbf{F}_0 sim(\Omega t+\varphi_0)$$

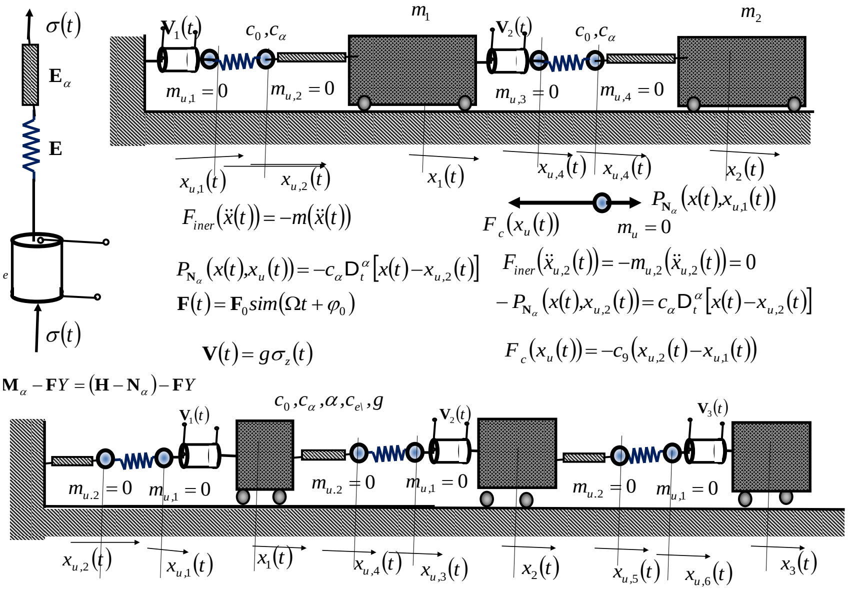
$$\mathbf{V}(t)=g\sigma_z(t) \qquad \mathbf{E}_k=\frac{1}{2}\sum_{i=1}^{i=4} m_i\dot{x}_i^2$$

$$\mathbf{E}_p=\frac{1}{2}(c_0+c_e)\left[(x_2-x_1)^2+(x_3-x_2)^2+(x_4-x_3)^2\right]$$

$$\mathbf{\Phi}_w=\frac{1}{2}c_\alpha \mathsf{D}_t^\alpha\left[(x_2-x_1)^2+(x_3-x_2)^2+(x_4-x_3)^2\right]$$

$$\mathbf{K}/\mathbf{FY}_\alpha=(\mathbf{H}/\mathbf{N}_\alpha)/\mathbf{FY}$$

Fig. 19. The rheological complex Kelvin - Voigt-Faraday's dynamic system-homogeneous chain of the oscillations, of the fractional type, piezoelectric properties, with finite number of degrees of freedom



$$F_{iner}(\ddot{x}(t))=-m(\ddot{x}(t))$$

$$P_{\mathbf{N}_\alpha}(x(t),x_u(t))=-c_\alpha \mathsf{D}_t^\alpha[x(t)-x_{u,2}(t)]$$

$$\mathbf{F}(t)=\mathbf{F}_0 sim(\Omega t+\varphi_0)$$

$$\mathbf{V}(t)=g\sigma_z(t)$$

$$F_{iner}(\ddot{x}_{u,2}(t))=-m_{u,2}(\ddot{x}_{u,2}(t))=0$$

$$-P_{\mathbf{N}_\alpha}(x(t),x_{u,2}(t))=c_\alpha \mathsf{D}_t^\alpha[x(t)-x_{u,2}(t)]$$

$$F_c(x_u(t))=-c_9(x_{u,2}(t)-x_{u,1}(t))$$

$$\mathbf{M}_\alpha-\mathbf{F}Y=(\mathbf{H}-\mathbf{N}_\alpha)-\mathbf{F}Y$$

Fig. 20. The rheological complex Maxwell-Faraday's dynamic system-homogeneous chain of the crawler type, of the fractional type, piezoelectric properties with finite number of degrees of freedom

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