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MODES OF NON-HOMOGENEOUS DAMPED BEAMS ON A WINKLER-TYPE ELASTIC LAYER

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Abstract. *The stepped beam considered here can be used to model many parts in mechanical assemblies, such as crankshafts, gearboxes, etc., where a beam is submerged into some medium. In such cases, the beam consists of two parts with different material properties (mass, stiffness). It is externally non-homogenously damped and it rests on a Winkler elastic layer. The elastic layer is represented by continuously distributed springs along the stepped beam. Using Newton's second law and classical elasticity theory, a system of partial differential equations of motion is derived. Different boundary conditions are applied. In order to determine eigenvalues and eigenvectors, we exploit the finite difference method for solving vibration problems of stepped beams. Results obtained using the finite difference method are compared with the analytical results, which are obtained using the Bernoulli-Fourier method. It is determined that the difference of the values obtained using two different methods is negligibly slight. For the analytical solution, the complete derivation of the characteristic equation for the clamped-clamped boundary conditions is given. For the other boundary conditions, (pinned-pinned, clamped-free, free-free) characteristic equations are given without derivation since the procedure is similar. Overdamped and underdamped vibration is investigated. The influence of the stiffness of the Winkler layer on Eigen characteristics is discussed.*

Key words: *Stepped beam, Damped beam, Vibration, Winkler layer, Finite difference method*

1. INTRODUCTION

Beam-like structures with stepwise changes in cross-section called stepped beams are widely used in the practice of civil and mechanical engineering and can also be used as a

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proper approximation of nonuniform beams. Therefore, dynamics of stepped beams is a problem of great importance. Stepped beam structures have found widespread application in various engineering fields, such as bridges, tall buildings, rotating machines, robotic arms, aerospace structures, etc. These beams can be used to model many rotational parts in mechanical assemblies, such as crankshafts, gearboxes, etc. They can be submerged on one side into a medium and this interaction can be modeled as an externally non-homogenously damped beam resting on a Winkler elastic layer. To determine the critical rotational speed of such elements, it is necessary to know critical resonant frequencies in order to prevent the undesired effects. These elements can be modeled as stepped beams. Analyzing the vibration behavior and stability, and determining resonant frequencies are important tasks of engineering mechanics, and they can be crucial in developing more reliable mechanical assemblies and machines. Besides axially rotating stepped beams, this model of a beam on an elastic layer can be used for modeling the behavior of a beam, which can be used in special purpose applications.

Kelly and Srinivas [1] and Srinivas [2] developed a general theory for determining natural frequencies and mode shapes for a system of elastically loaded Euler-Bernoulli beams. However, natural frequencies and vibration modes are determined using overly complicated numerical methods. The paper by Kelly and Nicely [3] provides exact solutions for a system of homogeneous Euler-Bernoulli beams interconnected with a Kelvin-Voigt layer. All the beams are of the same length and boundary conditions, but with different geometric shapes and material properties. This paper also provides a numerical example with a system of 5 cylindrical beams. Karličić *et al.* [4] studied axial vibration of systems of simply supported nanobeams interconnected with a Winkler elastic layer using the nonlocal Eringen continuum theory. De Rosa and Lippellio [5] applied the Euler-Bernoulli beam theory to analyze free vibration of single walled carbon nanotubes. Beams are immersed into an elastic medium described with Winkler and Pasternak models. They used Hamilton's principle for deriving system displacement equations and boundary conditions, which are solved with the Differential Quadrature Method (DQM). This paper also discusses different influences on the free vibration such as types of elastic medium coefficients, nonlocal parameters, types of supports. In the paper by Bakhtiari-Nejad [6] vibration analysis of piezo-actuated micro cantilever Euler-Bernoulli stepped beam is done and the influence of nonlocality and piezoelectric characteristics on resonance is discussed. Froio and Rizzi [7] gave an analytical solution for elastic bending of an Euler-Bernoulli beam laying on a variable nonlinear Winkler foundation. The papers by Jang and Bert [8, 9] present the exact solution for fundamental natural frequencies of the stepped beam for different boundary conditions. It is shown that frequency depends on the relation of steps. The paper by Jaworski and Dowell [10] theoretically and experimentally investigates transversal free vibration of multi-step cantilever beams with different cross-sections of each step. The authors notice considerable differences in natural frequencies obtained by analytical beam theory methods, multidimensional models of finite elements and experiments. Koplow *et al.* [11] obtained a closed form analytical solution for the dynamic response of a two-stepped beam. They applied free boundary conditions on both ends and determined frequent response functions when an external harmonic force is applied on one end. Lee and Bergman [12] proposed a method for determining eigenvalues for free and forced vibration of stepped beams and rectangular plates. Structures are divided into elementary substructures for which the authors determine a displacement field in the form of dynamic Green's functions**.** Naguleswaran [13] proposed an analytical method for calculating oscillation frequencies of stepped beams with 2 or 3 steps of different cross sections. He tested his method with 45 different supports where elastic connection supports were also considered. The same author [14] investigated vibration and stability of a stepped beam with 2 and 3 steps with different axial forces in each step. Step position, flexural rigidity, mass per unit length and axial force in degrees are taken as normalized system parameters. Tufekci and Yigit [15] investigated vibration of two-stepped circular beams and they applied an initial value method to obtain an exact analytical solution. Bagdatli et al. [16] obtained equations of motion by using Hamilton's principle and applied the Newton-Rapson method to obtain vibration frequency of a linearized nonlinear stepped beam, and then predicted nonlinear frequencies using artificial neural networks. Oniszczuk [17] analyzed free vibrations of two simply supported parallel beams with an elastic Winkler layer in between. The system motion was described with a homogeneous system of two partial differential equations and solved using the Bernoulli-Fourier method. He also determined the natural frequencies of the system vibration. Vu et al. [18] presented a method for solving a forced vibration system of two beams under the influence of a harmonic excitation force applied only on one beam. Beams are interconnected with a series of spring and damping elements. The solution is given in a closed form. Wang [19] investigated vibration of stepped beams on a Winkler layer and gave analytical and numerical solutions for the simplest model of a stepped beam. Thambiratnam [20] also solved free stepped beam oscillations but with finite elements. Dinckal [21] also applied the finite element method but for vibration of carbon nanotubes. Atanasov et al. [22] examined free vibration and buckling of an Euler-Bernoulli double-microbeam system under the compressive axial loading with a temperature change effect. The beams are joined by Pasternak's elastic layer. Karličić et al. [23] studied axial vibration of an elastic multi-nanorod system embedded in an elastic medium. They obtained natural frequencies using the finite difference method. Nešić et al. proposed a nonlocal (stress gradient) [24] and nonlocal strain gradient [25] fractional viscoelastic model of a nanobeam resting on the fractional viscoelastic foundation and under the influence of the transversal harmonic load. They gave a solution of the problem in the form of amplitude-frequency response curves based on the following methods: perturbation multiple scales, Newmark, and incremental harmonic balance combined with the continuation technique. Janevski et al. [26] investigated transverse vibration of a two-step Timoshenko beam under axial loading, and Boiangiu et al. [27] examined vibration of a multistep elastic Euler-Bernoulli beam experimentally. However, they did not consider damping and elastic layer as presented in this paper. Friswell and Lees [28] analytically computed eigenvalues and eigenfunctions of a two-step damped beam, however only with simply supported boundary conditions.

This paper presents two methodologies for computation of eigenvalues and eigenfunctions of an externally non-homogenously damped stepped beam oscillating on a Winkler elastic layer with various support types. The number of steps, i.e., homogenous segments could be chosen arbitrarily with the presented methods. However, for simplicity reasons, the examples studied in this paper have only two steps. The following boundary conditions are utilized: clamped-clamped, pinned-pinned, clamped-free, and free-free. The analytical methodology presented here is similar to the one given in [28], but our model is extended with a Winkler layer and more different support types are used. Finite difference is the chosen numerical method. Also, for the numerical example, the eigenvalues of oscillation and the first several modes are determined. The results using both finite difference and analytical method are validated with the paper [28]. We also analyze the influence of the layer stiffness on the vibration modes.

The motivation for this research is the very common application of structural elements, which can be modeled with stepped beams in practice. These elements often operate in an environment where a fluid is used to cool and lubricate them. The interaction between the fluid and the structure can be modeled as a damped beam resting on the Winkler layer. In a general case, a medium can be a fluid but also any other material. The purpose of this research is to present a model of a stepped shaft in a box submerged into a medium, as such elements can often be found in the engineering practice. Besides that, the aim is to present methods for obtaining eigenfrequencies of the beam and to pursue investigation of damped stepped beam vibration on a Winkler layer for overdamped and underdamped modes. Two methods are presented, the numerical and the analytical. The finite difference method is chosen as the numerical method and the solution of coupled differential beam equations with boundary conditions is explained in detail. The following chapters present methods for numerical and analytical determination of eigenvalues of an externally non-homogenously damped stepped beam oscillating on a Winkler layer with various support types.

This paper honors the late Professor Predrag Kozić†, who passed away during the publication process. He formulated the concept of this paper and was supervising the research.

2. Mathematical model and boundary conditions

Here we investigate the vibration of the Euler-Bernoulli stepped beam shown in Fig. 1. It consists of two parts, the first one of length L_1, with bending stiffness k_1, damping coefficient c_1, and mass per unit length m_1, and the second one of length L_2, with parameters k_2, c_2, m_2. The beam is supported on its ends by clamped supports and it oscillates on an elastic layer of stiffness K (Fig 1.a). A detailed derivation will be given for clamped-clamped boundary conditions, while only final equations and results will be provided for the rest, since the procedure is very similar.

In general, during the vibration of the stepped beam, work is produced by the external dynamic load, the internal forces, the damping forces and the internal cross-sectional forces.

The equilibrium conditions of the dynamic forces acting on an infinitesimal layered-beam element yield the following coupled governing differential equations of the system

$$k_1 \frac{\partial^4 w_1}{\partial x^4} + m_1 \frac{\partial^2 w_1}{\partial t^2} + c_1 \frac{\partial w_1}{\partial t} + K w_1 = 0, \quad 0 \le x \le L_1, \tag{1}$$

$$k_2 \frac{\partial^4 w_2}{\partial x^4} + m_2 \frac{\partial^2 w_2}{\partial t^2} + c_2 \frac{\partial w_2}{\partial t} + K w_2 = 0, \quad L_1 \le x \le L. \tag{2}$$

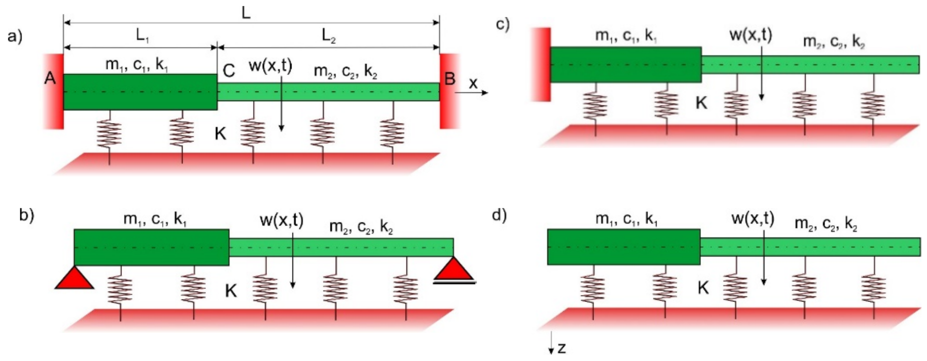

Fig. 1 Double-segment beam vibrating on Winkler layer under different boundary conditions: a) clamped-clamped, b) pinned-pinned, c) clamped-free, d) free-free

Boundary conditions for the clamped-clamped stepped beam: At points A and B the vertical displacement and rotation angle are zero:

$$w_1(x,t)\big|_{x=0} = w_2(x,t)\big|_{x=L} = 0,$$
$$\left.\frac{\partial w_1(x,t)}{\partial x}\right|_{x=0} = \left.\frac{\partial w_2(x,t)}{\partial x}\right|_{x=L} = 0. \tag{3}$$

Boundary conditions for the simple supported stepped beam: At points A and B the vertical displacement and moment are zero, i.e.

$$w_1(x,t)\big|_{x=0} = w_2(x,t)\big|_{x=L} = 0,$$
$$k_1\left.\frac{\partial^2 w_1(x,t)}{\partial x^2}\right|_{x=0} = k_2\left.\frac{\partial^2 w_2(x,t)}{\partial x^2}\right|_{x=L} = 0. \tag{4}$$

Boundary conditions for the clamped-free beam: At points A the vertical displacement and rotation angle are zero and at point B the transversal force and the moment are zero, i.e.

$$w_1(x,t)\big|_{x=0} = \left.\frac{\partial w_1(x,t)}{\partial x}\right|_{x=0} = 0,$$
$$k_2\left.\frac{\partial^2 w_2(x,t)}{\partial x^2}\right|_{x=L} = k_2\left.\frac{\partial^3 w_2(x,t)}{\partial x^3}\right|_{x=L} = 0. \tag{5}$$

Boundary conditions for the free-free beam: At points A and B the transversal force and moment are zero, i.e.

$$k_1 \frac{\partial^2 w_1(x,t)}{\partial x^2}\bigg|_{x=0} = k_1 \frac{\partial^3 w_1(x,t)}{\partial x^3}\bigg|_{x=0} = 0,$$
$$k_2 \frac{\partial^2 w_2(x,t)}{\partial x^2}\bigg|_{x=L} = k_2 \frac{\partial^3 w_2(x,t)}{\partial x^3}\bigg|_{x=L} = 0. \tag{6}$$

And at point C we should have continuity in the vertical deflection, slope, flexural moment and transversal force, i.e.

$$w_1(x,t)\big|_{x=L_1} = w_2(x,t)\big|_{x=L_1}, \qquad \frac{\partial w_1(x,t)}{\partial x}\bigg|_{x=L_1} = \frac{\partial w_2(x,t)}{\partial x}\bigg|_{x=L_1},$$
$$k_1 \frac{\partial^2 w_1(x,t)}{\partial x^2}\bigg|_{x=L_1} = k_2 \frac{\partial^2 w_2(x,t)}{\partial x^2}\bigg|_{x=L_1}, \qquad k_1 \frac{\partial^3 w_1(x,t)}{\partial x^2}\bigg|_{x=L_1} = k_2 \frac{\partial^3 w_2(x,t)}{\partial x^3}\bigg|_{x=L_1}. \tag{7}$$

3. NUMERICAL MODEL

The finite difference method is used to obtain eigenvalues of the oscillating two-segment beam on an elastic layer of the Winkler type under the different boundary conditions. The application of this method for solving the eigenvalue problem of stepped beams was inspired by Subrahmanyam [29], where finite difference method was used to solve vibration of uniform beams. But in this paper the finite difference methodology is extended to the analysis of stepped beam vibration on an elastic layer for different boundary conditions. We assume that the solution for displacement is a product of the amplitude function depending only on the coordinate x and the time function depending only on the time t, as written in the form

$$w_i(x,t) = X^{(i)}(x) \cdot T(t), \qquad i = 1,2. \tag{8}$$

Since time is unique for both beams we have $T_1(t)=T_2(t)=T(t)$. We assume the time function as

$$T(t) = e^{\lambda t}. \tag{9}$$

By introducing the time function from Eq. (9) into Eqs. (1) - (2), we obtain

$$k_i \frac{d^4 X^{(i)}}{dx^4} = -(m_i \lambda^2 + c_i \lambda + K) X^{(i)}, \qquad i = 1,2. \tag{10}$$

For the discretization of displacement derivatives, we use the following approximations

$$\frac{dX_j^{(i)}}{dx} = \frac{X_{j+1}^{(i)} - X_{j-1}^{(i)}}{2h}, \qquad j = 1,\ldots,n_i, \tag{11}$$

$$\frac{d^2X_j^{(i)}}{dx^2}=\frac{X_{j-1}^{(i)}-2X_j^{(i)}+X_{j+1}^{(i)}}{h^2},\qquad j=1,\dots,n_i, \tag{12}$$

$$\frac{d^3X_j^{(i)}}{dx^3}=\frac{-X_{j-2}^{(i)}+2X_{j-1}^{(i)}-2X_{j+1}^{(i)}+X_{j+2}^{(i)}}{2h^3},\qquad j=1,\dots,n_i, \tag{13}$$

$$\frac{d^4X_j^{(i)}}{dx^4}=\frac{X_{j-2}^{(i)}-4X_{j-1}^{(i)}+6X_j^{(i)}-4X_{j+1}^{(i)}+X_{j+2}^{(i)}}{h^4},\qquad j=1,\dots,n_i. \tag{14}$$

where h is the distance between two discretization points. We assume that this length is constant throughout the beam for simpler calculations.

For the approximation of displacement derivatives, it is also possible to use higher order approximations which are more complicated and require more points than Eqs. 10-14, but they do not provide much better results.

For the case when two beam segments are equal, the distance between two discretization points can be obtained as $h=L/n$, where $L=L_1+L_2$ is the total length of the beam, and n is the number of discretized elements of equal lengths into which the beam is divided.

More generally, for the case when two beam segments are not equal, we can proceed in the following way. The maximal possible distance between two discretization points h_{max} can be obtained as the greatest common divisor of the values of the lengths of two segments: $h_{max}=GCD(L_1,L_2)$ (GCD stands for greatest common divisor). The minimal total number of discretized steps, and the minimal numbers of discretized steps for the beams 1 and 2 are respectively $n_{min}=L/h_{max}$, $n_{1min}=L_1/h_{max}$, $n_{2min}=L_2/h_{max}$.

The total number of steps for the whole double beam and the number of steps for beam 1 and 2 are $n_{tot}=nn_{min}$, $n_1=nn_{1min}$, $n_2=nn_{2min}$. The n is a natural number. When n is increased, the mesh becomes finer, results closer to real, but the computational time also increases. The discretization step length can now be obtained as $h=L/n_{tot}$.

By substituting h and Eq. (14) in Eq. (10), we obtain

$$\frac{k_i}{h^4}\left(X_{j-2}^{(i)}-4X_{j-1}^{(i)}+6X_j^{(i)}-4X_{j+1}^{(i)}+X_{j+2}^{(i)}\right)=-\left(m_i\lambda^2+c_i\lambda+K\right)X_j^{(i)},\quad i=1,12,2,\quad j=1,\dots,n. \tag{15}$$

which can be written in a more descriptive form as

$$\begin{aligned}
&\frac{k_1}{h^4}\left(X_{j-2}^{(1)}-4X_{j-1}^{(1)}+6X_j^{(1)}-4X_{j+1}^{(1)}+X_{j+2}^{(1)}\right)=-\left(m_1\lambda^2+c_1\lambda+K\right)X_j^{(1)},\qquad j=1,\dots,n_1-1,\\
&\frac{k_{12}}{h^4}\left(X_{j-2}^{(12)}-4X_{j-1}^{(12)}+6X_j^{(12)}-4X_{j+1}^{(12)}+X_{j+2}^{(12)}\right)=-\left(m_{12}\lambda^2+c_{12}\lambda+K\right)X_j^{(12)},\qquad j=n_1,\\
&\frac{k_2}{h^4}\left(X_{j-2}^{(2)}-4X_{j-1}^{(2)}+6X_j^{(2)}-4X_{j+1}^{(2)}+X_{j+2}^{(2)}\right)=-\left(m_2\lambda^2+c_2\lambda+K\right)X_j^{(2)},\qquad j=n_1+1,\dots,n_1+n_2.
\end{aligned} \tag{16}$$

Indexes i=1,2 correspond to beam segments and index i=12 corresponds to common point from both beam segments.

After joining the previous three equations as

$$\frac{k_i}{h^4}\left(X_{j-2} - 4X_{j-1} + 6X_j - 4X_{j+1} + X_{j+2}\right) = -\left(m_i\lambda^2 + c_i\lambda + K\right)X_j, \quad i = 1,12,2, \quad j = 1,\dots,n. \tag{17}$$

where in the connecting point of the two beams we average the values of mass, stiffness and damping $k_{12}=(k_1+k_2)/2$, $m_{12}=(m_1+m_2)/2$, $c_{12}=(c_1+c_2)/2$.

There are also other ways to treat the connecting point of the two beams [30], but this averaging procedure gives satisfying results.

By applying the clamped-clamped boundary conditions from Eq. (3) to Eq. (17), we obtain the following system of linear algebraic equations:

$$\frac{k_1}{h^4}\left(7X_1 - 4X_2 + X_3\right) = -\left(m_1\lambda^2 + c_1\lambda + K\right)X_1,$$

$$\frac{k_1}{h^4}\left(4X_1 + 6X_2 - 4X_3 + X_4\right) = -\left(m_1\lambda^2 + c_1\lambda + K\right)X_2,$$

$$\frac{k_1}{h^4}\left(X_1 - 4X_2 + 6X_3 - 4X_4 + X_5\right) = -\left(m_1\lambda^2 + c_1\lambda + K\right)X_3,$$

$$\frac{k_1}{h^4}\left(X_2 - 4X_3 + 6X_4 - 4X_5 + X_6\right) = -\left(m_1\lambda^2 + c_1\lambda + K\right)X_4,$$

....

$$\frac{k_1}{h^4}\left(X_{j-3} - 4X_{j-2} + 6X_{j-1} - 4X_j + X_{j+1}\right) = -\left(m_1\lambda^2 + c_1\lambda + K\right)X_{j-1},$$

$$\frac{k_{12}}{h^4}\left(X_{j-2} - 4X_{j-1} + 6X_j - 4X_{j+1} + X_{j+2}\right) = -\left(m_{12}\lambda^2 + c_{12}\lambda + K\right)X_j,$$

$$\frac{k_2}{h^4}\left(X_{j-1} - 4X_j + 6X_{j+1} - 4X_{j+2} + X_{j+3}\right) = -\left(m_2\lambda^2 + c_2\lambda + K\right)X_{j+1},$$

....

$$\frac{k_2}{h^4}\left(X_{n-5} - 4X_{n-4} + 6X_{n-3} - 4X_{n-2} + X_{n-1}\right) = -\left(m_2\lambda^2 + c_2\lambda + K\right)X_{n-3},$$

$$\frac{k_2}{h^4}\left(X_{n-4} - 4X_{n-3} + 6X_{n-2} - 4X_{n-1}\right) = -\left(m_2\lambda^2 + c_2\lambda + K\right)X_{n-2},$$

$$\frac{k_2}{h^4}\left(X_{n-3} - 4X_{n-2} + 6X_{n-1}\right) = -\left(m_2\lambda^2 + c_2\lambda + K\right)X_{n-1}. \tag{18}$$

Eigenvalues are computed when the determinant of system Eq. (24) vanishes. The determinant has the following form

$$\det\begin{bmatrix} 7k_1+h^4\alpha_1 & -4k_1 & k_1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ -4k_1 & 6k_1+h^4\alpha_1 & -4k_1 & k_1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ k_1 & -4k_1 & 6k_1+h^4\alpha_1 & -4k_1 & k_1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & k_1 & -4k_1 & 6k_1+h^4\alpha_1 & -4k_1 & k_1 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & k_{12} & -4k_{12} & 6k_{12}+h^4\alpha_{12} & -4k_{12} & k_{12} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & k_2 & -4k_2 & 6k_2+h^4\alpha_2 & -4k_2 & k_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & k_2 & -4k_2 & 6k_2+h^4\alpha_2 & -4k_2 & k_2 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & k_2 & -4k_2 & 6k_2+h^4\alpha_2 & -4k_2 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & k_2 & -4k_2 & 7k_2+h^4\alpha_2 \end{bmatrix}=0, \quad (19)$$

where $\alpha_i=m_i\lambda^2+c_i\lambda+K$, i=1,12,2. By finding the zeros of the determinant (Eq. 19), we obtain eigenvalues λ. For other boundary conditions the procedure is the same, and only values at the first and the last two nodes are different. For the pinned-pinned boundary conditions (Fig. 1(b)), the frequency equation is obtained by replacing 7 with 5 in the first and last main diagonal terms in the matrix (Eq. 19). For the clamped-free boundary conditions (Fig. 1(c)), the last two discrete equations are as in Eqs. (20) and others are the same as in Eqs. (18).

....

$$\frac{k_2}{h^4}\left(X_{n-4}-4X_{n-3}+5X_{n-2}-2X_{n-1}\right)=-\left(m_2\lambda^2+c_2\lambda+K\right)X_{n-2},$$

$$\frac{k_2}{h^4}\left(2X_{n-3}-4X_{n-2}+2X_{n-1}\right)=-\left(m_2\lambda^2+c_2\lambda+K\right)X_{n-1}. \quad (20)$$

In this case, we have one extra equation than in the previous cases.

For the free-free boundary conditions (Fig. 1(d)), the equations are the same as in the case with the clamped-clamped boundary conditions (Eqs. 18), but with two extra equations. The only difference is in the first two and the last two equations, which are in this case given in Eqs. (21).

$$\frac{k_1}{h^4}\left(2X_1-4X_2+2X_3\right)=-\left(m_1\lambda^2+c_1\lambda+K\right)X_1,$$

$$\frac{k_1}{h^4}\left(-2X_1+5X_2-4X_3+X_4\right)=-\left(m_1\lambda^2+c_1\lambda+K\right)X_2,$$

....

$$\frac{k_2}{h^4}\left(X_{n-4}-4X_{n-3}+5X_{n-2}-2X_{n-1}\right)=-\left(m_2\lambda^2+c_2\lambda+K\right)X_{n-2},$$

$$\frac{k_2}{h^4}\left(2X_{n-3}-4X_{n-2}+2X_{n-1}\right)=-\left(m_2\lambda^2+c_2\lambda+K\right)X_{n-1}. \quad (21)$$

4. ANALYTICAL METHOD

The Bernoulli-Fourier method that separates mutually independent variables is used for solving differential Eqs. (1, 2). Using this method, we assume that the solution for displacement is a product of the amplitude function depending only on the coordinate x and the time function depending only on the time t, as written in the form

$$w_i(x,t) = X_i(x)\cdot T(t), \qquad i=1,2. \tag{22}$$

Since time is unique for both beams we have $T_1(t)=T_2(t)=T(t)$.

By substituting Eq. (22) and corresponding derivatives in differential Eqs. (1, 2), we obtain the following relations

$$k_1 X_1^{IV}(x)T(t) + m_1 X_1(x)\ddot{T}(t) + c_1 X_1(x)\dot{T}(t) + K X_1(x)T(t) = 0, \quad 0 \le x \le L_1, \tag{23}$$

$$k_2 X_2^{IV}(x)T(t) + m_2 X_2(x)\ddot{T}(t) + c_2 X_2(x)\dot{T}(t) + K X_2(x)T(t) = 0, \quad L_1 \le x \le L. \tag{24}$$

Dividing both equations with $m_i X_i(x) T(t)$, i=1,2 and after some simple algebraic steps, a system of differential equations is obtained in the symmetric form

$$\frac{k_i}{m_i}\frac{X_i^{IV}(x)}{X_i(x)} = -\frac{\ddot{T}(t) + \frac{c_i}{m_i}\dot{T}(t) + \frac{K}{m_i}T(t)}{T(t)} = \kappa_i, \qquad i=1,2. \tag{25}$$

where κ_i is the constant which can be a positive, negative or complex number.

From Eq. (25) we get the following two couples of single variable differential equations

$$X_1^{IV}(x) - \frac{m_1}{k_1}\kappa_1 X_1(x) = 0, \qquad 0 \le x \le L_1, \tag{26}$$

$$X_2^{IV}(x) - \frac{m_2}{k_2}\kappa_2 X_2(x) = 0, \qquad L_1 \le x \le L, \tag{27}$$

$$\ddot{T}(t) + \frac{c_1}{m_1}\dot{T}(t) + \left(\frac{K}{m_1} + \kappa_1\right)T(t) = 0, \tag{28}$$

$$\ddot{T}(t) + \frac{c_2}{m_2}\dot{T}(t) + \left(\frac{K}{m_2} + \kappa_2\right)T(t) = 0, \tag{29}$$

which are solved afterwards.

5. MODES

We determine κ_i from Eqs. (28, 29). The time function for overdamped modes is assumed as

$$T(t) = e^{-\lambda t}, \qquad \lambda > 0. \tag{30}$$

By substituting the assumed solution Eq. (30) in Eqs. (28, 29), we obtain characteristic equations. After simple term rearrangements, from these characteristic equations constant $\varkappa_i$ κ_i can be expressed as

$$\kappa_i = \frac{c_i}{m_i}\lambda - \lambda^2 - \frac{K}{m_i}\cdot \tag{31}$$

5.1. Case 1 ($\kappa_i > 0, \kappa_i \in R$)

When $\varkappa_i > 0$ $\kappa_i > 0, \kappa_i \in R$ for i=1,2 and by introducing the next substitution

$$\mu_i^4 = \frac{m_i}{k_i}\kappa_i, \qquad i = 1,2, \tag{32}$$

into Eqs. (26, 27) we obtain

$$X_1(x) = A_1(\sin\mu_1 x - \sinh\mu_1 x) + C_1(\cos\mu_1 x - \cosh\mu_1 x), \qquad 0 \le x \le L_1, \tag{33}$$

$$X_2(x) = A_1[\sin\mu_2(L-x) - \sinh\mu_2(L-x)] + C_2[\cos\mu_2(L-x) - \cosh\mu_2(L-x)], \quad L_1 \le x \le L. \tag{34}$$

By substituting Eqs. (33)-(34) in boundary conditions Eq. (7), we obtain a system of algebraic equations for unknown constants A_1, C_1, A_2 and C_2:

$$\begin{bmatrix} s_1 - sh_1 & c_1 - ch_1 & -(s_2 - sh_2) & -(c_2 - ch_2) \\ \mu_1(c_1 - ch_1) & -\mu_1(s_1 + sh_1) & \mu_2(c_2 - ch_2) & -\mu_2(s_2 + sh_2) \\ -k_1\mu_1^2(s_1 + sh_1) & -k_1\mu_1^2(c_1 + ch_1) & k_2\mu_2^2(s_2 + sh_2) & k_2\mu_2^2(c_2 + ch_2) \\ -k_1\mu_1^3(c_1 + ch_1) & k_1\mu_1^3(s_1 - sh_1) & k_2\mu_2^3(c_2 + ch_2) & k_2\mu_2^3(s_2 - sh_2) \end{bmatrix} \begin{bmatrix} A_1 \\ C_1 \\ A_2 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{35}$$

where for the sake of simplification the following notations are introduced

$$\begin{aligned} s_1 &= \sin\mu_1 L_1, & s_2 &= \sin\mu_2 L_2, & c_1 &= \cos\mu_1 L_1, & c_2 &= \cos\mu_2 L_2, \\ sh_1 &= \sinh\mu_1 L_1, & sh_2 &= \sinh\mu_2 L_2, & ch_1 &= \cosh\mu_1 L_1, & ch_2 &= \cosh\mu_2 L_2. \end{aligned} \tag{36}$$

The system of algebraic Eqs. (35) has nontrivial solutions for constants A_1, C_1, A_2 and C_2, when the determinant of system Eq. (35) is equal to zero. By introducing Eqs. (31, 32) into system Eq. (35), we obtain the equation for the unknown λ as $f_\lambda(\mu_1(\lambda),\mu_1(\lambda))=f(\lambda)=0$.

It can be observed that to obtain λ_j, j=1,2,…,∞, we calculate μ_1 and μ_2 from Eqs. (31, 32) in the form

$$\mu_i(\lambda_j) = \pm\sqrt{\pm\sqrt{\frac{c_i}{k_i}\lambda_j - \frac{m_i}{k_i}\lambda_j^2 - \frac{K}{k_i}}} \quad i = 1,2, \quad j = 1,2,\ldots,\infty\cdot \tag{37}$$

For the obtained λ_j, j=1,2,…,∞, the system of Eqs. (35) has an infinite number of solutions for unknown constants A_1, C_1, A_2 and C_2. They cannot be explicitly determined, but their relations can as

$$\frac{A_1^{(j)}}{K_{41}^{(\lambda_j)}} = \frac{C_1^{(j)}}{K_{42}^{(\lambda_j)}} = \frac{A_2^{(j)}}{K_{43}^{(\lambda_j)}} = \frac{C_2^{(j)}}{K_{44}^{(\lambda_j)}} = C_{\lambda_j}\,, \tag{38}$$

where $C_{\lambda j}$ is an arbitrary constant.

By substituting constants $A_1^{(j)}$, $C_1^{(j)}$, $A_2^{(j)}$ and $C_2^{(j)}$ in Eqs. (33, 34), one can obtain amplitude functions as

$$X_1(x) = K_{41}^{(\lambda_j)} C_{\lambda_j} \left(\sin\mu_1 x - \sinh\mu_1 x\right) + K_{42}^{(\lambda_j)} C_{\lambda_j} \left(\cos\mu_1 x - \cosh\mu_1 x\right), \quad 0 \le x \le L_1, \tag{39}$$

$$X_2(x) = K_{43}^{(\lambda_j)} C_{\lambda_j} \left[\sin\mu_2(L-x) - \sinh\mu_2(L-x)\right] + K_{44}^{(\lambda_j)} C_{\lambda_j} \left[\cos\mu_2(L-x) - \cosh\mu_2(L-x)\right], \quad L_1 \le x \le L. \tag{40}$$

Bending displacement of the two beam portions $z_i(x,t)$ is determined from Eqs. (22) and (30) as

$$W_i(x,t) = \mathrm{Re}\left[w_i(x,t)\right] = \mathrm{Re}\left[X_i(x)e^{-\lambda t}\right], \qquad i = 1,2. \tag{41}$$

5.2. Case 2 ($\kappa_i < 0, \kappa_i \in R$)

For the case when $\varkappa_i > 0$ $\kappa_i < 0, \kappa_i \in R$ for i=1,2, by introducing the following substitution

$$\eta_i^4 = -\frac{m_i}{4k_i}\kappa_i > 0, \qquad i = 1,2, \tag{42}$$

in Eqs. (26, 27), we obtain

$$X_1^{IV}(x) + 4\eta_1^4 X_1(x) = 0, \qquad 0 \le x \le L_1, \tag{43}$$

$$X_2^{IV}(x) + 4\eta_2^4 X_2(x) = 0, \qquad L_1 \le x \le L. \tag{44}$$

After solving the system of differential Eqs. (43, 44), we obtain

$$X_1(x) = A_1\left[\cos\eta_1 x \sin\eta_1 x - \sinh\mu_1 x \cos\eta_1 x\right] + C_1 \sinh\mu_1 x \sin\eta_1 x, \qquad 0 \le x \le L_1, \tag{45}$$

$$X_2(x) = A_2\left[\cos\eta_2(L-x)\sin\eta_2(L-x) - \sinh\mu_2(L-x)\cos\eta_2(L-x)\right] + C_2 \sinh\mu_2(L-x)\sin\eta_2(L-x), \quad L_1 \le x \le L. \tag{46}$$

By substituting Eqs. (45, 46) in the boundary conditions (Eq. 7), we obtain a system of algebraic equations for unknown constants A_1, C_1, A_2 and C_2

$$\begin{bmatrix} ch_1s_1 - sh_1c_1 & sh_1s_1 & sh_2c_2 - ch_2s_2 & -sh_2s_2 \\ 2\eta_1 sh_1s_1 & \eta_1(ch_1s_1 + sh_1c_1) & 2\eta_2 sh_2s_2 & \eta_2(ch_2s_2 + sh_2c_2) \\ 2\eta_1^2 k_1(ch_1s_1 + sh_1c_1) & 2\eta_1^2 k_1 ch_1c_1 & -2\eta_2^2 k_2(ch_2s_2 + sh_2c_2) & -2\eta_2^2 k_2 ch_2c_2 \\ 4\eta_1^3 k_1 ch_1c_1 & 2\eta_1^3 k_1(sh_1c_1 - ch_1s_1) & 4\eta_2^3 k_2 ch_2c_2 & 2\eta_2^3 k_2(sh_2c_2 - ch_2s_2) \end{bmatrix} \begin{bmatrix} A_1 \\ C_1 \\ A_2 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{47}$$

where the following notations are introduced

$$\begin{aligned} s_1 &= \sin\eta_1 L_1, \quad & s_2 &= \sin\eta_2 L_2, \quad & c_1 &= \cos\eta_1 L_1, \quad & c_2 &= \cos\eta_2 L_2, \\ sh_1 &= \sinh\eta_1 L_1, \quad & sh_2 &= \sinh\eta_2 L_2, \quad & ch_1 &= \cosh\eta_1 L_1, \quad & ch_2 &= \cosh\eta_2 L_2. \end{aligned} \tag{48}$$

Bending displacement $W_i(x,t)$ can be determined in a similar way as in case 1 from Eq. (41).

5.3. Case 3 ($\kappa_i \in C$)

This case is similar to case 1. We determine κ_i from the time function in Eqs. (28, 29). The time function is now for underdamped modes assumed as

$$T(t) = e^{\lambda t}, \qquad \lambda > 0. \tag{49}$$

Constant κ_i is complex and can be expressed with Eq. (35). In the system of differential Eqs. (28, 29), we use substitution

$$\nu_i^4 = \frac{m_i}{k_i}\kappa_i, \qquad i = 1,2, \tag{50}$$

and after following the same procedure as in case 1, we obtain the following system of algebraic equations:

$$\begin{bmatrix} s_1 - sh_1 & c_1 - ch_1 & -(s_2 - sh_2) & -(c_2 - ch_2) \\ \nu_1(c_1 - ch_1) & -\nu_1(s_1 + sh_1) & \nu_2(c_2 - ch_2) & -\nu_2(s_2 + sh_2) \\ -k_1\nu_1^2(s_1 + sh_1) & -k_1\nu_1^2(c_1 + ch_1) & k_2\nu_2^2(s_2 + sh_2) & k_2\nu_2^2(c_2 + ch_2) \\ -k_1\nu_1^3(c_1 + ch_1) & k_1\nu_1^3(s_1 - sh_1) & k_2\nu_2^3(c_2 + ch_2) & k_2\nu_2^3(s_2 - sh_2) \end{bmatrix} \begin{bmatrix} A_1 \\ C_1 \\ A_2 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{51}$$

where

$$\begin{aligned} s_1 &= \sin\nu_1 L_1, \quad & s_2 &= \sin\nu_2 L_2, \quad & c_1 &= \cos\nu_1 L_1, \quad & c_2 &= \cos\nu_2 L_2, \\ sh_1 &= \sinh\nu_1 L_1, \quad & sh_2 &= \sinh\nu_2 L_2, \quad & ch_1 &= \cosh\nu_1 L_1, \quad & ch_2 &= \cosh\nu_2 L_2. \end{aligned} \tag{52}$$

Bending displacement $W_i(x,t)$ can be determined in a similar way as in case 1 from Eq. (41).

Following the same procedure as in the previous cases but using boundary conditions for pinned-pinned, clamped-free and free-free beams from Eqs. (4-6), eigenvalues and bending displacements are obtained. The frequency equation for the pinned-pinned boundary conditions is

$$\det\begin{bmatrix} sh_1 & sh_1 & -sh_2 & -sh_2 \\ \mu_1 ch_1 & \mu_1 ch_1 & \mu_2 ch_2 & \mu_2 ch_2 \\ k_1\mu_1^2 sh_1 & k_1\mu_1^2 sh_1 & -k_2\mu_2^2 sh_2 & -k_2\mu_2^2 sh_2 \\ k_1\mu_1^3 ch_1 & k_1\mu_1^3 ch_1 & k_2\mu_2^3 ch_2 & k_2\mu_2^3 ch_2 \end{bmatrix} = 0. \tag{53}$$

For the clamped-free boundary condition the frequency equation is

$$\det\begin{bmatrix} s_1 - sh_1 & c_1 - ch_1 & -(s_2 + sh_2) & -(c_2 + ch_2) \\ \mu_1(c_1 - ch_1) & -\mu_1(s_1 + sh_1) & \mu_2(c_2 + ch_2) & -\mu_2(s_2 - sh_2) \\ -k_1\mu_1^2(s_1 + sh_1) & -k_1\mu_1^2(c_1 + ch_1) & k_2\mu_2^2(s_2 - sh_2) & k_2\mu_2^2(c_2 - ch_2) \\ -k_1\mu_1^3(c_1 + ch_1) & k_1\mu_1^3(s_1 - sh_1) & -k_2\mu_2^3(c_2 - ch_2) & k_2\mu_2^3(s_2 + sh_2) \end{bmatrix} = 0. \quad (54)$$

For the free-free boundary condition the frequency equation is

$$\det\begin{bmatrix} s_1 + sh_1 & c_1 + ch_1 & -(s_2 + sh_2) & -(c_2 + ch_2) \\ \mu_1(c_1 + ch_1) & -\mu_1(s_1 - sh_1) & \mu_2(c_2 + ch_2) & -\mu_2(s_2 - sh_2) \\ -k_1\mu_1^2(s_1 - sh_1) & -k_1\mu_1^2(c_1 - ch_1) & k_2\mu_2^2(s_2 - sh_2) & k_2\mu_2^2(c_2 - ch_2) \\ -k_1\mu_1^3(c_1 - ch_1) & k_1\mu_1^3(s_1 + sh_1) & -k_2\mu_2^3(c_2 - ch_2) & k_2\mu_2^3(s_2 + sh_2) \end{bmatrix} = 0. \quad (55)$$

In Eqs. (53-55) we used substitutions from Eq. (36). Bending displacement $W_i(x,t)$ can be determined in a similar way as in Eq. (41).

6. NUMERICAL EXAMPLE

The proposed method is tested with numerical values as in the paper by Friswell and Lees [28]. They are given in Table 1, where index 1 represents the beam AC, and index 2 the beam CB. The obtained eigenvalues for the first five modes and different stiffnesses of the Winkler layer for different boundary conditions are given in Tables 2-9. The overdamped modes appeared in case 2, and both overdamped and underdamped in case 1. For the obtained eigenvalues, corresponding eigenfunctions are obtained. In Tables 2-9 we can see the influence of the support stiffness on eigenvalues.

Table 1 Numerical values for material and geometry parameters of a stepped beam

Beam AC	Beam CB	
	Case 1	Case 2
L_1=1 m m_1=10 kg/m c_1=0 k_1=100 N/m^2	L_2=2 m m_2=20 kg/m c_2=100 N s/m^2 k_2=100 N/m^2	L_2=2 m m_2=40 kg/m c_2=10 kN s/m^2 k_2=100 N/m^2

Clamped-clamped boundary conditions are applied and obtained eigenvalues are represented in Tables 2-4. Finite difference results obtained with 90, 210 and 300 discretization steps are compared to the analytical results. The value for the stiffness of the elastic layer is varied and the influence on vibration modes can be seen in these tables.

In Table 2 eigenvalues are given for the overdamped modes for case 2, while Table 3 provides eigenvalues for the underdamped modes for case 1.

Table 2 Eigenvalues for case 2 for the first five overdamped modes, beam with CLAMPED-CLAMPED boundary conditions, finite difference and analytical results

CC Case 2	Analytical	FD 90 steps	FD 210 steps	FD 300 steps
Mode		K = 1 N/m		
1	-0.0004	-0.0711	-0.0711	-0.0711
2	-0.7104	-0.7085	-0.7098	-0.7116
3	-3.5312	-3.5141	-3.5278	-3.5294
4	-12.0457	-11.9400	-12.0260	-12.0358
5	-35.0867	-34.5202	-34.9812	-35.0347
Mode		K = 50 N/m		
1	-0.0200	-0.0767	-0.0767	-0.0767
2	-0.7322	-0.7155	-0.7168	-0.7170
3	-3.5554	-3.5230	-3.5367	-3.5383
4	-12.0741	-11.9516	-12.0375	-12.0474
5	-35.1244	-34.5351	-34.9962	-35.0497
Mode		K = 1000 N/m		
1	-0.4842	-0.1833	-0.1834	-0.1834
2	-1.1538	-0.8495	-0.8508	-0.8509
3	-4.0216	-3.6933	-3.7071	-3.7087
4	-12.6221	-12.1732	-12.2596	-12.2696
5	-35.8535	-34.8224	-35.2847	-35.3384

Table 3 Eigenvalues for the first five underdamped modes, beam with CLAMPED-CLAMPED boundary conditions for case 1

CC Case 1	Analytical	FD 90 steps	FD 210 steps	FD 300 steps
Mode		K = 1 N/m		
1	-2.330 ± 5.279 j	-2.330 ± 5.272 j	-2.330 ± 5.274 j	-2.330 ± 5.274 j
2	-1.974 ± 16.788 j	-1.974 ± 16.768 j	-1.974 ± 16.784 j	-1.974 ± 16.785 j
3	-1.648 ± 33.975 j	-1.649 ± 33.906 j	-1.649 ± 33.962 j	-1.648 ± 33.968 j
4	-1.682 ± 55.115 j	-1.683 ± 54.956 j	-1.682 ± 55.086 j	-1.682 ± 55.100 j
5	-1.875 ± 82.052 j	-1.874 ± 81.718 j	-1.875 ± 81.990 j	-1.875 ± 82.021 j
Mode		K = 1000 N/m		
1	-2.331 ± 11.312 j	-2.348 ± 9.002 j	-2.348 ± 9.003 j	-2.348 ± 9.003 j
2	-1.974 ± 19.541 j	-1.984 ± 18.483 j	-1.984 ± 18.497 j	-1.984 ± 18.499 j
3	-1.648 ± 35.413 j	-1.640 ± 34.881 j	-1.639 ± 34.936 j	-1.639 ± 34.942 j
4	-1.682 ± 56.013 j	-1.670 ± 55.556 j	-1.670 ± 55.684 j	-1.669 ± 55.699 j
5	-1.875 ± 82.658 j	-1.871 ± 82.099 j	-1.872 ± 82.370 j	-1.872 ± 82.401 j

Table 4 Eigenvalues for the first five underdamped modes of the beam with CLAMPED-CLAMPED boundary conditions for case 2

CC Case 2	Analytical	FD 90 steps	FD 210 steps	FD 300 steps
Mode	K = 1 N/m			
1	-8.023 ± 52.442 j	-8.070 ± 52.394 j	-8.031 ± 52.433 j	-8.027 ± 52.437 j
2	-23.168 ± 156.770 j	-23.240 ± 156.013 j	-23.181 ± 156.629 j	-23.174 ± 156.701 j
3	-116.365 ± 45.674 j	-116.451 ± 42.153 j	-116.374 ± 45.042 j	-116.365 ± 45.365 j
4	-113.527 ± 131.119 j	-113.702 ± 128.121 j	-113.559 ± 130.567 j	-113.543 ± 130.848 j
5	-111.558 ± 201.453 j	-111.734 ± 197.172 j	-111.591 ± 200.664 j	-111.574 ± 201.066 j
Mode	K = 1000 N/m			
1	-7.821 ± 53.419 j	-7.888 ± 53.390 j	-7.850 ± 53.427 j	-7.846 ± 53.432 j
2	-23.100 ± 157.086 j	-23.179 ± 156.351 j	-23.120 ± 156.965 j	-23.113 ± 157.037 j
3	-116.401 ± 46.855 j	-116.493 ± 42.531 j	-116.416 ± 45.394 j	-116.406 ± 45.715 j
4	-113.555 ± 131.522 j	-113.729 ± 128.230 j	-113.586 ± 130.674 j	-113.569 ± 130.955 j
5	-111.574 ± 201.707 j	-111.747 ± 197.238 j	-111.603 ± 200.729 j	-111.586 ± 201.131 j

Table 5 Eigenvalues for case 2 for the first five overdamped modes, beam with PINNED-PINNED boundary conditions

PP Case 2	Analytical	FD 90 steps	FD 210 steps	FD 300 steps
Mode	K = 1 N/m			
1	-0.0004	-0.0150	-0.0150	-0.0150
2	-0.3369	-0.3362	-0.3365	-0.3365
3	-2.2127	-2.2059	-2.2112	-2.2118
4	-8.5804	-8.5272	-8.5703	-8.5753
5	-26.5250	-26.2114	-26.4666	-26.4961
Mode	K = 50 N/m			
1	-0.0200	-0.0209	-0.0209	-0.0209
2	-0.3599	-0.3445	-0.3448	-0.3448
3	-2.2391	-2.2171	-2.2225	-2.2231
4	-8.6121	-8.5425	-8.5857	-8.5907
5	-26.5630	-26.2290	-26.4842	-26.5137
Mode	K = 1000 N/m			
1	-0.4006	-0.1312	-0.1312	-0.1312
2	-0.8009	-0.4986	-0.4989	-0.4990
3	-2.7431	-2.4282	-2.4337	-2.4344
4	-9.2165	-8.8290	-8.8727	-8.8777
5	-27.2980	-26.5604	-26.8158	-26.8453

Table 4 provides eigenvalues of underdamped modes for case 2. In Tables 5-7 pinned-pinned boundary conditions are applied and obtained eigenvalues are given. Same analysis as for the clamped-clamped case is used. Again, the finite difference results obtained with 90, 210 and 300 discretization steps are compared to the analytical results. Table 5 provides the first five eigenvalues for the overdamped modes for case 2, Table 6 provides eigenvalues for the underdamped modes for case 1, while Table 7 provides eigenvalues for underdamped modes for case 2.

Table 6 Eigenvalues for the first five underdamped modes, beam with PINNED-PINNED boundary conditions for case 1

PP Case 1	Analytical	FD 90 steps	FD 210 steps	FD 300 steps
Mode	K = 1 N/m			
1	-2.255 ± 1.310 j	-2.255 ± 1.292 j	-2.255 ± 1.293 j	-2.255 ± 1.293 j
2	-1.794 ± 10.908 j	-1.794 ± 10.901 j	-1.794 ± 10.905 j	-1.794 ± 10.905 j
3	-1.574 ± 24.865 j	-1.575 ± 24.840 j	-1.574 ± 24.860 j	-1.574 ± 24.862 j
4	-1.788 ± 43.166 j	-1.788 ± 43.098 j	-1.788 ± 43.153 j	-1.788 ± 43.160 j
5	-1.878 ± 68.118 j	-1.878 ± 67.940 j	-1.878 ± 68.085 j	-1.878 ± 68.102 j
Mode	K = 1000 N/m			
1	-2.259 ± 10.115 j	-2.317 ± 7.493 j	-2.317 ± 7.493 j	-2.317 ± 7.493 j
2	-1.790 ± 14.794 j	-1.799 ± 13.532 j	-1.799 ± 13.535 j	-1.799 ± 13.535 j
3	-1.574 ± 26.791 j	-1.533 ± 26.186 j	-1.532 ± 26.206 j	-1.532 ± 26.208 j
4	-1.788 ± 44.306 j	-1.769 ± 43.836 j	-1.769 ± 43.890 j	-1.769 ± 43.897 j
5	-1.878 ± 68.848 j	-1.877 ± 68.398 j	-1.877 ± 68.542 j	-1.877 ± 68.558 j

Table 7 Eigenvalues for the first five underdamped modes, beam with PINNED-PINNED boundary conditions for case 2

PP Case 2	Analytical	FD 90 steps	FD 210 steps	FD 300 steps
Mode	K = 1 N/m			
1	-5.593 ± 35.045 j	-5.633 ± 35.061 j	-5.600 ± 35.048 j	-5.596 ± 35.047 j
2	-18.528 ± 124.767 j	-18.663 ± 124.516 j	-18.553 ± 124.720 j	-18.528 ± 124.767 j
3	-39.370 ± 278.170 j	-39.603 ± 276.295 j	-39.410 ± 277.824 j	-39.390 ± 278.003 j
4	-114.301 ± 112.950 j	-114.409 ± 110.489 j	-114.317 ± 112.500 j	-114.306 ± 112.731 j
5	-112.104 ± 183.550 j	-112.268 ± 180.118 j	-112.131 ± 182.920 j	-112.115 ± 183.242 j
Mode	K = 1000 N/m			
1	-5.295 ± 36.504 j	-5.357 ± 36.535 j	-5.325 ± 36.522 j	-5.321 ± 36.520 j
2	-18.441 ± 125.168 j	-18.587 ± 124.939 j	-18.477 ± 125.142 j	-18.464 ± 125.166 j
3	-39.340 ± 278.350 j	-39.566 ± 276.485 j	-39.374 ± 278.012 j	-39.353 ± 278.191 j
4	-114.330 ± 113.421 j	-114.438 ± 110.618 j	-114.346 ± 112.626 j	-114.335 ± 112.856 j
5	-112.120 ± 183.830 j	-112.283 ± 180.194 j	-112.146 ± 182.994 j	-112.130 ± 183.316 j

In Table 8 obtained eigenvalues are given for the first five underdamped modes for the vibration of the beam with clamped-free boundary conditions for case 1, using finite difference and analytical method. Finally, in Table 9 obtained eigenvalues are given, using finite difference and analytical method, for the first five underdamped modes for the vibration of the beam with free-free boundary conditions for case 1. Stiffness of the layer is also varied.

Table 8 Eigenvalues for the first five underdamped modes, beam with CLAMPED-FREE boundary conditions for case 1

CF Case 1	Analytical	FD 90 steps	FD 210 steps	FD 300 steps
Mode	K = 1 N/m			
1	-2.341 ± 5.163 j	-2.341 ± 5.155 j	-2.341 ± 5.158 j	-2.341 ± 5.158 j
2	-1.775 ± 116.284 j	-1.776 ± 115.621 j	-1.775 ± 116.162 j	-1.775 ± 116.224 j
3	-1.681 ± 302.619 j	-1.688 ± 298.819 j	-1.682 ± 302.127 j	-1.682 ± 302.510 j
4	-1.844 ± 247.076 j	-1.845 ± 244.319 j	-1.844 ± 246.566 j	-1.844 ± 246.826 j
5	-1.856 ± 426.403 j	-1.855 ± 418.596 j	-1.856 ± 424.955 j	-1.856 ± 425.693 j
Mode	K = 1000 N/m			
1	-2.491 ± 9.726 j	-2.356 ± 8.924 j	-2.356 ± 8.925 j	-2.356 ± 8.925 j
2	-1.775 ± 116.713 j	-1.776 ± 115.899 j	-1.775 ± 116.439 j	-1.775 ± 116.501 j
3	-1.681 ± 303.043 j	-1.688 ± 298.929 j	-1.688 ± 302.237 j	-1.681 ± 302.619 j
4	-1.844 ± 247.278 j	-1.845 ± 244.448 j	-1.844 ± 246.694 j	-1.844 ± 246.954 j
5	-1.856 ± 426.520 j	-1.855 ± 418.671 j	-1.856 ± 425.029 j	-1.856 ± 425.767 j

Table 9 Eigenvalues for the first five underdamped modes, beam with FREE-FREE boundary conditions for case 1

FF Case 1	Analytical	FD 90 steps	FD 210 steps	FD 300 steps
Mode	K = 1 N/m			
1	-1.532 ± 5.668 j	-1.530 ± 5.663 j	-1.530 ± 5.665 j	-1.530 ± 5.665 j
2	-1.841 ± 16.988 j	-1.841 ± 16.969 j	-1.841 ± 16.984 j	-1.841 ± 16.985 j
3	-1.635 ± 33.961 j	-1.636 ± 33.893 j	-1.635 ± 33.948 j	-1.635 ± 33.954 j
4	-1.694 ± 55.040 j	-1.695 ± 54.881 j	-1.695 ± 55.010 j	-1.695 ± 55.025 j
5	-1.881 ± 82.016 j	-1.880 ± 81.682 j	-1.881 ± 81.954 j	-1.881 ± 81.986 j
Mode	K = 1000 N/m			
1	-1.968 ± 11.080 j	-2.121 ± 8.357 j	-2.121 ± 8.358 j	-2.121 ± 8.358 j
2	-1.843 ± 19.704 j	-1.812 ± 18.730 j	-1.813 ± 18.744 j	-1.813 ± 18.746 j
3	-1.635 ± 35.400 j	-1.625 ± 34.872 j	-1.623 ± 34.926 j	-1.623 ± 34.932 j
4	-1.694 ± 55.939 j	-1.682 ± 55.479 j	-1.682 ± 55.607 j	-1.682 ± 55.622 j
5	-1.881 ± 82.623 j	-1.878 ± 82.063 j	-1.878 ± 82.334 j	-1.878 ± 82.365 j

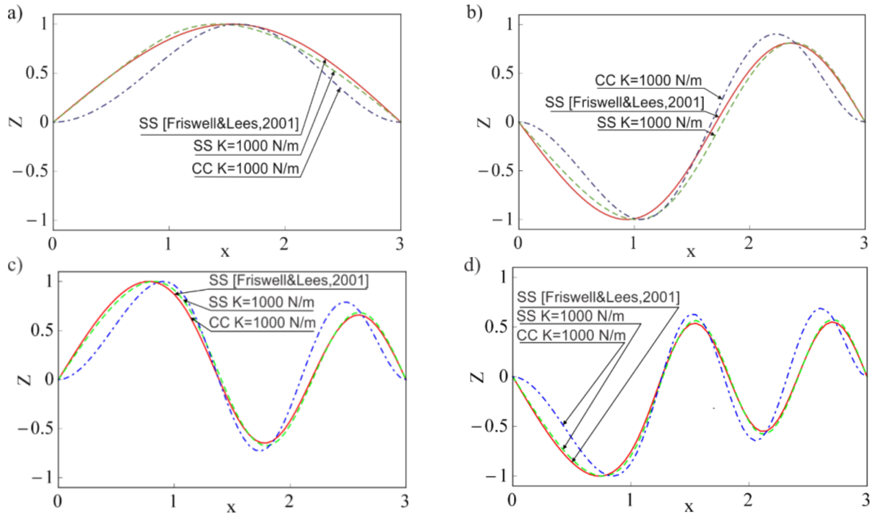

Fig. 2 The first four overdamped vibration modes for case 2 (——simply supported, K=0 N/m [28]; - - - simply supported, K=1000 N/m; -·- clamped-clamped, K=1000 N/m)

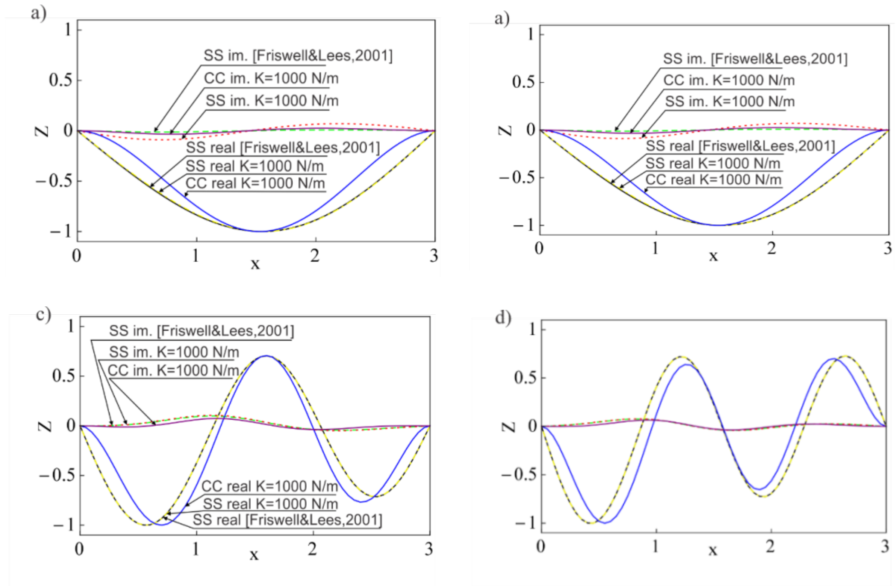

Fig. 3 The first four underdamped vibration modes for case 1 (for K=0N/m, [28]): —— simply supported real part, - - - simply supported imaginary part; for K=1000 N/m: - · simply supported real part, ······ simply supported imaginary part, —— clamped-clamped real part, —— clamped-clamped imaginary part)

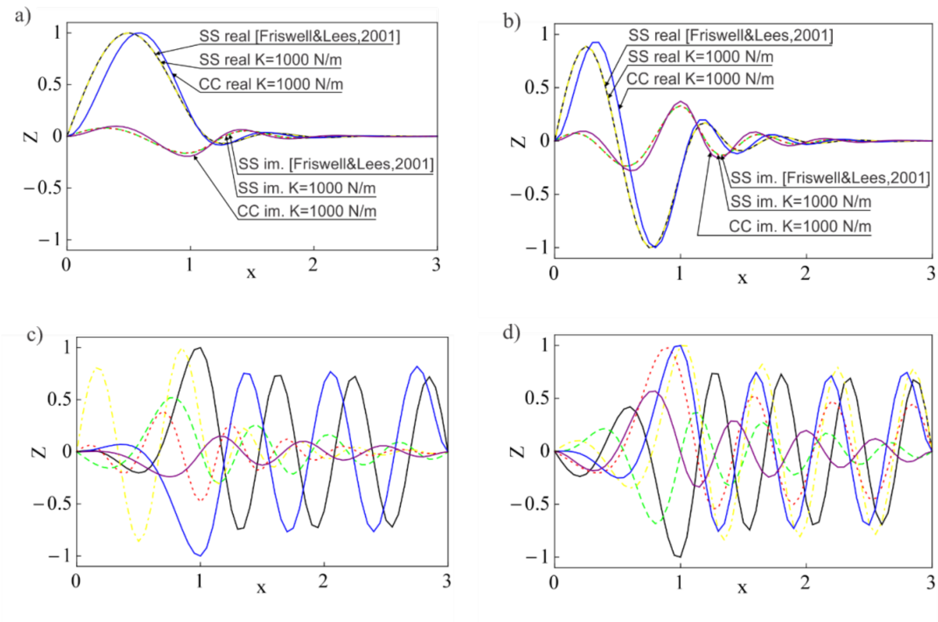

Fig. 4 The first four underdamped vibration modes for case 2 (for K=0N/m, Friswell & Lees 2001): —— simply supported real part, - - - simply supported imaginary part; for K=1000 N/m: - · - simply supported real part, ········ simply supported imaginary part, —— clamped-clamped real part, —— clamped-clamped imaginary part)

Using the obtained eigenvalues for K=1000 N/m from Tables 2-7, and eigenvalues from the paper [28] which can be obtained from our equations by putting K=0 N/m for a simply supported two-step beam, we determine the vibration modes. The real eigenvalues give us the overdamped modes in Fig. (2), and the conjugate complex eigenvalues give us the underdamped modes in Figs. (3, 4). Displacement, which is represented on the vertical axis, is normalized. The horizontal axis corresponds to the position of the point on the longest beam axis. Figs. (3, 4) show real and imaginary parts of the modal displacement of the beams with clamped-clamped and pinned-pinned boundary conditions.

Comparing the normalized modes (overdamped and underdamped) with the results obtained in the paper [28] with a beam without the layer (K=0) for pinned-pinned boundary conditions, we observe similarities in the mode shape. As the stiffness of the layer approaches zero, the obtained eigenvalues for the beam with pinned-pinned boundary conditions are closer to the eigenvalues from [28]. For the underdamped vibration, the amplitude of the imaginary part is smaller than the amplitude of the real part. We also observe in Figs. (2-4) that there are better similarities in some mode shapes for certain cases. By comparing the simply supported beams with K=0 N/m and K=1000 N/m in Figs. (2, 3), one can see that for the underdamped vibration (Fig. 3), the amplitude curves are much closer to one another than for the overdamped vibration (Fig. 2). Also, the layer influence is higher for case 2 parameters and overdamped vibration.

In Tables 2-7, we can observe that all solutions are stable, because real parts of eigenvalues are negative. For case 2 parameters, the imaginary part of eigenvalues is zero, which implies that modes are overdamped. In general, the real part of an eigenvalue is a measure of the decay rate, and the imaginary part is a measure of the frequency.

The advantage of the finite difference method is more visible when the number of segments (change of cross-section) is much higher than two, which was our case. For a beam with n segments, using the analytical method, eigenvalues are obtained by finding zeros of the determinant of the order 4 (n - 1), which is hard to determine for n >> 2.

7. CONCLUSIONS

This paper presents two methods - finite difference and analytical method for the calculation of eigenvalues and eigenfunctions of a continuous damped beam, consisting of two parts, which oscillates on an elastic support of the Winkler type, accompanied by a numerical example. By using each of the methods, we derived characteristic equations for the clamped-clamped boundary conditions and obtained eigenvalues. For the other boundary conditions such as pinned-pinned, clamped-free and free-free we provide only final characteristic equations, because the procedure is similar.

For the finite difference method beams are discretized with 90, 210 and 300 steps in total. Eigenvalues obtained with the finer mesh are closer to analytical results than eigenvalues with the coarse mesh. Computation time of eigenvalues with a 300step mesh is approximately 50% of the time the analytical method uses, and the finite difference method with 90 steps uses around 5% of the analytical method time. Finding zeros with analytical methods involves many trigonometric and hyperbolic functions, while we only have polynomial functions with finite differences. Therefore, the analytical method is more precise but slower than the numerical one with a reasonably small step size. The precision as well as the computation time are increasing in the finite difference method by reducing the step size. From Tables 2-9, we can observe the convergence of the numerical method, as with an increasing number of discretization points, numerical results are closer to results obtained analytically. Numerical results with 300 steps are the closest to analytical results and frequencies obtained with 90 steps are the furthest from the analytical ones.

The advantage of the finite difference method is more visible when the number of segments (change of cross-section) is much higher than two, which was our case. For a beam with n segments, using the analytical method, eigenvalues are obtained by finding zeros of the determinant of the order 4 (n - 1), which is hard to determine for n >> 2.

The influence of the elastic layer stiffness on vibration was also investigated. From Tables 2-9 we can conclude that by increasing the layer stiffness, the absolute value of the eigenvalue increases and therefore the frequency too. This result corresponds to the real physical phenomena. Eigenfrequency of the beam oscillating on a stiffer support is higher than on a support with smaller stiffness.

Depending on the material parameters and the geometry of the beam, we have overdamped or underdamped modes. For underdamped vibration, the amplitude of the imaginary part is smaller than the amplitude of the real part. The results are also compared with pinned-pinned boundary conditions and the results from the literature [28], and they show good agreement. The results obtained with finite differences for a two-segment beam with clamped-clamped, pinned-pinned, clamped-free and free-free boundary conditions are also validated with the analytical method. This paper also provides an analytical procedure for solving vibration problems of a 2-segment stepped beam. We can see good agreement of our results with the results from the literature and between the numerical and analytical results.

The numerical finite difference method with the methodology presented in this paper could be applied for obtaining vibrational frequency of more complex and precise beam models. In this paper, we used the Euler-Bernoulli beam model where it is assumed that the cross-section remains undeformed and perpendicular to the elastic line in the longitudinal direction. For higher beam theories such as Timoshenko or Reddy-Bickford, which also include shear deformation, more accurate results are expected, but equations of motion become more complex. It is hard to solve them analytically with the presented analytical method, but with the finite difference method it is possible to obtain the frequency values. Effects of shear deformation on the free vibration of homogeneus elastic beams with general boundary conditions are investigated by Li and Hua [31]. In our case of a stepped elastic beam on a Winkler layer we would expect a similar relative value of frequencies between different higher beam theories if we would include shear deformation in our model. This could be the task of some future study.

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