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MODELING AND SIMULATION OF STOCHASTIC PROCESSES

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Abstract. *This work presents a brief study of wideband stochastic processes. Models of wideband noises, white noise, real noise, and bound noise processes are analyzed in detail. After a deep analytical study, these processes are further simulated on a suggested simple model. For this purpose, a simple elastic nanobeam model is used. The model presents a nanobeam axially compressed on its ends, where the compression forces are simulated according to their definitions.*

Key words: *Modeling, Simulation, Wideband processes, Nanobeam, Axial compression*

1. Introduction

Stochastic processes have been greatly studied in recent years, but they are still attracting a lot of attention due their wide application in various fields such as physics, mathematics, finance, and engineering. The notion of stochastic processes is very important both in mathematical theory and its applications since it is used to model many various phenomena where the quantity of interest unpredictably varies discretely or continuously through time. A stochastic or random process is a mathematical object usually defined as a family of random variables that are indexed by some mathematical set, usually viewed as points in time, giving the interpretation of a stochastic process representing numerical values of some system randomly changing over time [1-2].

In many engineering applications, the excitations of dynamical systems can be described as wideband random processes. Gaussian white noise, real noise and bound noise processes are typical wideband stochastic processes. Gaussian white noise is a stationary and ergodic random process with zero mean that is defined by the following fundamental property: any two values of Gaussian white noise are statistically independent no matter how close they are in time and can be formally expressed as the time derivative of a Wiener or Brownian process whose roots date back to early papers of Albert Einstein, e.g. [3]. While the mathematical properties of white noise have been studied extensively and utilized in many fields, Gaussian white noise signal is not physically realizable and is only

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an idealization of practical excitations [4]. Another important wideband noise is called real noise, which is also called the Ornstein-Uhlenbeck process. The Ornstein-Uhlenbeck process has a long history in physics. First introduced in 1908 by Langevin [5] in his paper on Brownian motion, the process received a more thorough mathematical examination several decades later by Uhlenbeck and Ornstein [6], Chandrasekhar [7], and Wang and Uhlenbeck [8], and is today used as a standard term in the literature. The Ornstein-Uhlenbeck process is a simple, Gaussian, explicitly representable stationary process that is often used to model a realizable noise process, and as a result it is called a real noise process. Lately, a lot of researchers have been focused on another important class of stochastic processes: bounded noises. The bounded noise process was first engaged by Stratonovich [9] and has since been applied in certain engineering applications. The rise of scientific interest in bounded noises is motivated by the fact that in many applications stochastic processes can be inadequate mathematical models of the physical world because they are unbounded [10].

For simulations of these processes, a simple model of a nanobeam axially compressed on its ends is used. Based on the nonlocal theory of Eringen [11], nanobeam structures are widely studied by many researchers. For the first time, nonlocal theory was applied in nanotechnology by Peddieson, Buchanan and McNitt [12]. The nonlocal Euler–Bernoulli beam model was used in Nejad and Hadi [13] and Tuna and Kirca [14] for deriving the partial differential equation of transverse motion of a nanobeam. Reddy [15] and Reddy and Pang [16] proposed detailed studies on the application of nonlocal theory to bending, vibration and stability of nanobeam structures. The static and dynamic behavior of nanobeams was analyzed based on the nonlocal Euler–Bernoulli, Timoshenko, Reddy, Levinson and Aydogdu beam theories and presented in Aydogdu [17]. The influence of the nonlocal effects and length of a nanobeam on the natural frequencies, deflection and critical load was investigated in detail for each considered model. I. R. Pavlović et al. [18] studied the stochastic stability problem of a multi-nanobeam system subjected to compressive axial forces.

Using the knowledge from previous literature, this paper is focused on the simulation of an axially compressed nanobeam. In section 2 this paper starts with a discussion of stochastic processes, and then a real noise and bound process are modeled and simulated using MATLAB software in section 3. Section 4 ends the paper with concluding remarks.

2. STOCHASTIC PROCESSES

This section presents a detailed study of wideband stochastic processes. A random process can be defined as a continuous physical process caused by nondeterministic influences. In each of the series of experiments, a random process $X(t)$ generates a record $X_k(t)$, which represents the realization of an instance of the process function in an individual experiment. The random nature of the process is reflected in the fact that no two records are identical in every respect, as shown in Figure 1. In engineering, science, and economics there are many time-dependent random phenomena which can be modeled by stochastic processes. Often excitations of dynamical systems can be described as wideband stochastic processes with relatively flat power spectral density functions over a large frequency range. The equations of motion of many physical systems under the excitations of wideband stochastic processes can be approximated by Stratonovich stochastic differential equations.

Gaussian white noise processes, real noise processes and bound noise processes are typical wideband stochastic processes.

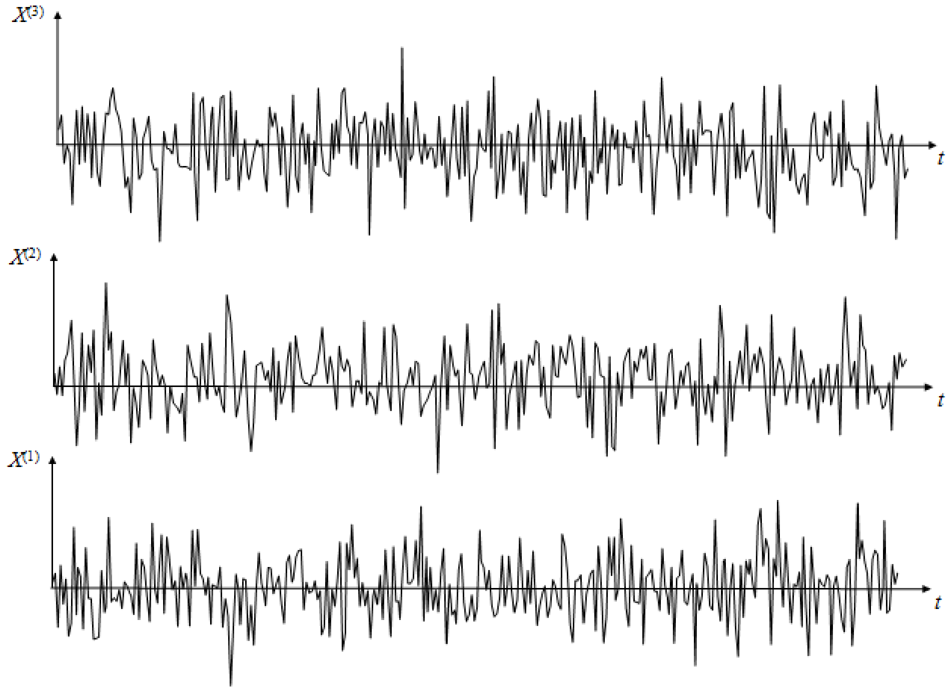

Fig. 1 Realizations of a stochastic process

Gaussian white noise is likely the most common stochastic model used in engineering applications. The power spectrum of GWN is constant over all frequencies, hence the name "white noise", in analogy to the white light that contains all (visible) wave lengths with the same power. A stochastic process $X(t)$ is said to be Gaussian white noise if $X(t)$ is normally distributed for each t and values $X(t_1)$ and $X(t_2)$ are independent for $t_1 \neq t_2$. The first assumption refers to the "Gaussian" and the second one to the "white". Clearly, this is a very simple model: it merely represents the drawing of independent normal random variables at different time instants. Still, it can be used to model complex stochastic systems.

White noise is mathematically defined as a stationary process X(t), $-\infty < t < +\infty$ with zero mean $E[X(t)] = 0$, whose autocorrelation is

$$E[X(t)X(t+\tau)] = R(\tau) = S_0\delta(\tau), \tag{1}$$

where $E[\cdot]$ denotes the expectation of $(\cdot)$, $\delta(t)$ is the Dirac delta function, and $\tau = t_2 - t_1$. The power spectral density $S(\omega)$ of $X(t)$ is constant over all frequencies

$$S(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega\tau} S_0 \delta(t)\, d\tau = S_0 = Constant. \quad (2)$$

The variance of a white noise process is $E[X^2(t)] = R(0) = \infty$. The formal definition of a white noise process is established to relate a white noise process with the Wiener process in the following form

$$X(t) = \sigma \dot{W}(t), \quad \sigma = \sqrt{S_0}, \quad (3)$$

where $W(t)$ is a standard Wiener process, $X(t)$ is then called the Gaussian white noise process, and σ is the standard deviation and is a measure of the spread of the probability density curve about the mean. A white noise process presents a stochastic excitation that is characterized by having equal intensity at different frequencies, thus not physically realizable and only an idealization of practical excitations.

As mentioned before, another important wideband stochastic process is the Ornstein-Uhlenbeck process or real noise process. Its original application was as a model for the velocity of tiny particles in liquids and gases to overcome unrealistic infinitely large velocity in a Brownian motion process. An Ornstein-Uhlenbeck process is used to model a realizable noise process and satisfies the Itô stochastic differential equation

$$dX(t) = -\alpha X(t)dt + \sigma dW(t), \quad X(t_0) = X_0, \quad (4)$$

where α is the positive constant. The solution to the stochastic differential equation (4) is

$$X(t) = X_0 e^{-\alpha(t-t_0)} + \sigma \int_{t_0}^{t} e^{-\alpha(t-s)} dW(s). \quad (5)$$

The expected value of $X(t)$ is

$$m_X(t) = E[X(t)] = e^{-\alpha(t-t_0)} E[X_0]. \quad (6)$$

The covariance function is defined by equation

$$K(t_1, t_2) = E[\{X(t_1) - m_X(t_1)\}\{X(t_2) - m_X(t_2)\}], \quad (7)$$

when $t_1 = t_2 = t$ we have the variance of $X(t)$ in the following form

$$Var[X(t)] = E[\{X(t) - m_X(t)\}^2] = K(t,t)$$

$$e^{-2\alpha(t-t_0)} Var(X_0) + \frac{\sigma^2}{2\alpha}\left[1 - e^{-2\alpha(t-t_0)}\right]. \quad (8)$$

By using the initial condition $X(t_0) = X_0$, from equation (6) and (8), $X(t)$ is a random variable with mean $\mu_{X(t)} = X_0 e^{-\alpha(t-t_0)}$ and standard deviation $\sigma_{X(t)}$ where

$$\sigma_{X(t)}^2 = \frac{\sigma^2}{2\alpha}\left[1 - e^{-2\alpha(t-t_0)}\right].$$

By determining the transition probability $q(X, t | X_0, t_0)$, it can be shown that $X(t)$, for given $X(t_0) = X_0$, is a normally distributed random variable. Further, from equations (6) and (8) we can conclude that for the arbitrary initial condition X_0

$$E[X(t)] \to 0, \; Var[X(t)] \to \frac{\sigma^2}{2\alpha}, \quad as\, t \to \infty, \quad (9)$$

which means that if the initial condition X_0 is normally distributed with zero mean and variance $\frac{\sigma^2}{2\alpha}$, than $X(t)$ is a stationary Gaussian process with the mean, $E[X(t)] = 0$, and the autocorrelation function in the exponential form $R(\tau) = E[X(t)X(t+\tau)] = \frac{\sigma^2}{2\alpha}e^{-\alpha|\tau|}$. The power spectral density function is defined by

$$S(\omega) = \int_{-\infty}^{+\infty} R(\tau)e^{-i\omega\tau}d\tau = \frac{\sigma^2}{\alpha^2+\omega^2} = \frac{\sigma^2}{\alpha^2}\frac{1}{1+\left(\frac{\omega}{\alpha}\right)^2}. \tag{10}$$

It is found that the parameter α characterizes the bandwidth of the process and $\left(\frac{\sigma}{\alpha}\right)^2$ shows the spectral density of the process. In the case when $\sigma = \alpha\sqrt{S_0} \to \infty$ the real noise process is reduced to a Gaussian white noise with the constant spectral density $S(\omega) = S_0$. Eqs. (4), (5), (10) provide a statistically complete description of the time evolution of the Ornstein-Uhlenbeck process.

Since the previously defined Gaussian white noise and Ornstein-Uhlenbeck processes are not bounded and may be an inadequate mathematical model of the physical world, not considering the bounded nature of stochastic fluctuations may lead to unrealistic model-based inferences. To avoid these problems, some stochastic models should be built on bounded noises. A bounded noise process is often defined as

$$Y(t) = cos[\nu t + \sigma W(t) + \theta], \tag{11}$$

where θ denotes a uniformly distributed random number in $(0,2\pi)$, and ν is the positive constant. The inclusion of the phase angle θ in equation (11) makes $Y(t)$ a stationary process. Eq. (11) can also be represented as

$$Y(t) = cos\, Z(t), \quad dZ(t) = \nu dt + \sigma dW(t), \tag{12}$$

with the initial condition $Z(0) = \theta$. The correlation function of $Y(t)$is

$$R(\tau) = E[Y(t)Y(t+\tau)] = \frac{1}{2}E[cos\{\nu t - \sigma W(t) + \sigma W(t+\tau)\}] +$$

$$+\frac{1}{2}E[cos\{\nu(2t+\tau) + \sigma W(t) + \sigma W(t+\tau) + 2\theta\}]. \tag{13}$$

If $\zeta(t,\tau) = \nu(2t+\tau) + \sigma W(t) + \sigma W(t+\tau)$, then the second expected value, which is the ensemble average at time t for fixed value of τ, is

$$E[\cos\{\zeta(t,\tau) + 2\theta\}] = E[\cos\zeta(t,\tau)]E[\cos 2\theta] - E[\sin\zeta(t,\tau)]E[\sin 2\theta] = 0. \tag{14}$$

Now the correlation function is

$$\frac{1}{2}\int_{-\infty}^{+\infty} \cos(\nu\tau + \sigma z)\cdot\frac{1}{\sqrt{2\pi|\tau|}}exp\left(-\frac{z^2}{2|\tau|}\right)dz = \frac{1}{2}\cos\nu\tau\, exp\left(-\frac{\sigma^2}{2}|\tau|\right), \tag{15}$$

where $z = [W(t+\tau) - W(t)]$ is a normally distributed random variable with zero mean and variance $|\tau|$. The spectral density function of $Y(t)$ is

$$S(\omega) = \int_{-\infty}^{+\infty} R(\tau)e^{-i\omega\tau}d\tau = \frac{\sigma^2\left(\omega^2+\nu^2+\frac{1}{4}\sigma^4\right)}{2\left[(\omega-\nu)^2+\frac{1}{4}\sigma^4\right]\left[(\omega+\nu)^2+\frac{1}{4}\sigma^4\right]}. \tag{16}$$

The mean-square value of the bounded noise process is fixed at

$$E[Y^2(t)] = R(0) = \frac{1}{2}. \tag{17}$$

When σ is small, bounded noise can be used to model a narrow-band process about frequency ν. When σ rises, the power density curve becomes flat, and bounded noise approximates a white noise process for large values of σ.

3. Example of a Simple Nanobeam

In this section a real noise and a bound process, as the most complex of the previously considered stochastic processes, are modeled and simulated using MATLAB software. The influence of these processes on dynamic system behavior is further presented on a simple axially compressed nanobeam.

Now, according to the work of Gillespie [19] and using the MATLAB program code for real noise process generation [20], the 10 seconds simulation of this process is given in Fig. 2. The jagged curves are composed of unconnected dots that give the values of the processes at each time step.

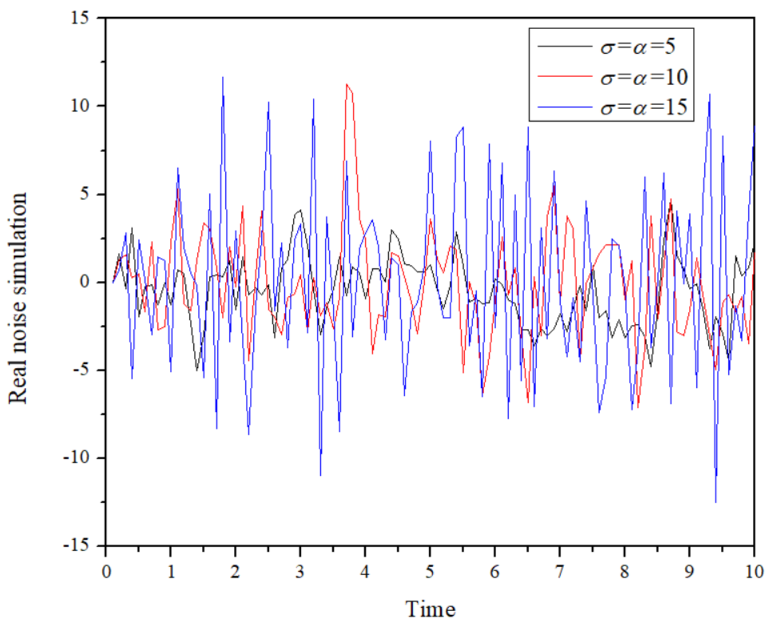

Fig. 2 Real noise process simulation in the function of parameters σ and α

The nanobeam model is studied in detail in many of the previous authors' works. For example, according to [18], the discretized form of the equation of motion of the nanobeam in the first oscillation mode is given by

$$\ddot{q} + 2\beta\varepsilon\dot{q} + \eta^2 q - \sqrt{\varepsilon} f(t)\pi^2 q = 0, \tag{18}$$

where ε is the small fluctuation parameter, β is the viscous damping coefficient, $f(t)$ presents the time-dependent stochastic function and

$$\eta^2 = \frac{\pi^4}{1+\mu^2\pi^2}. \tag{19}$$

According to equation (18) the following simulation scheme is suggested (Fig.3)

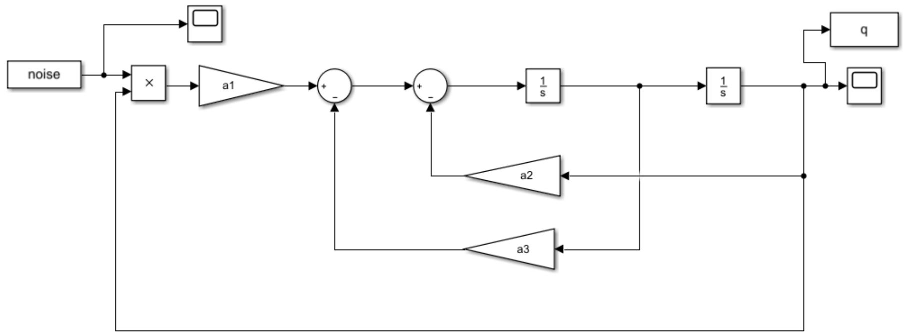

Fig. 3 Simulation scheme according to equation (18)

where a1=$\sqrt{\varepsilon}\pi^2$, a2=η^2, a3=$2\beta\varepsilon$, and where block noise presents the input signal in this model.

Fig. 4 shows the results of a simulation run of the nanobeam response under the real noise process for different values of the nonlocal parameter μ. At the beginning of the simulation, for t <2s, the nanobeam response does not vary much with the change of the parameter μ, while after 2s, as the time increases, so does the difference in the response of the beam due to the parameter change. It can also be seen that although the displacement range of the nanobeam decreases over time for all parameters μ, the largest change in the range occurs for the smallest values of the parameter μ.

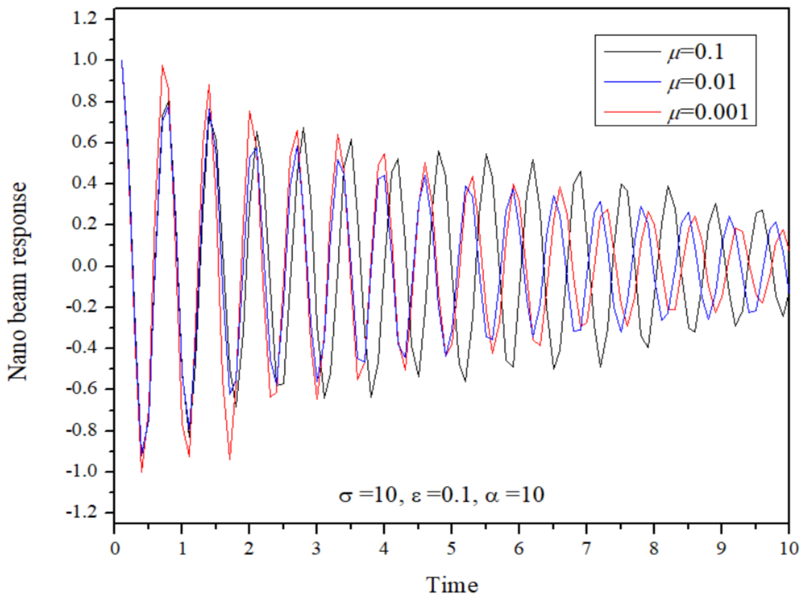

Fig. 4 Nanobeam response under the real noise process for different values of nonlocal parameter μ

Finally, according to its definition (11), a much simpler bound noise process is simulated and presented in Fig 5. After applying this process as the input parameter in model (18), the following stochastic response of the nanobeam is acquired (Fig. 6). As in Fig. 4, at the beginning the response of the nanobeam varies slightly with the change of the parameter μ, while with the increase in time t the differences in the response of the beam increase between the bound noise process with the parameter $\mu = 0.1$ and $\mu = 0.01$, while with a further decrease in the parameter μ the response of the beam changes insignificantly, which is shown in Fig. 6 by matching the lines describing the process with $\mu = 0.01$ and $\mu = 0.001$.

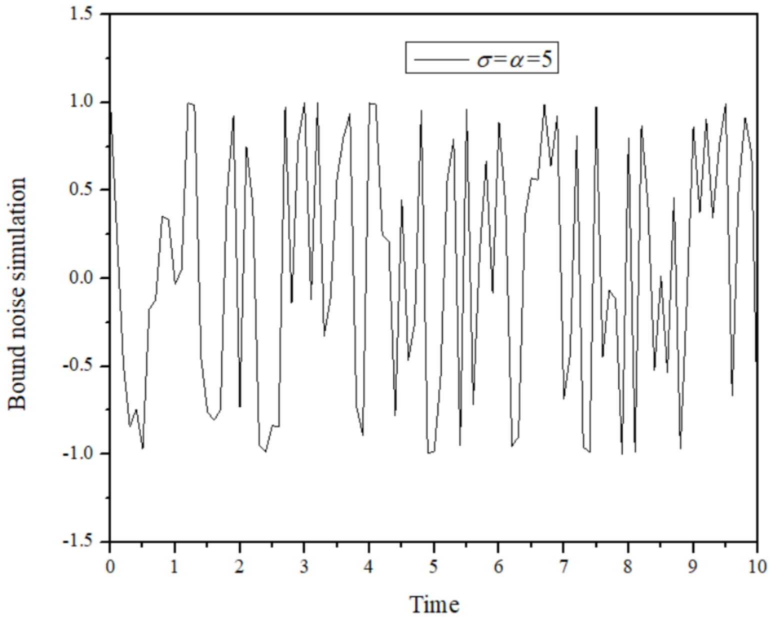

Fig. 5 Bound noise process simulation

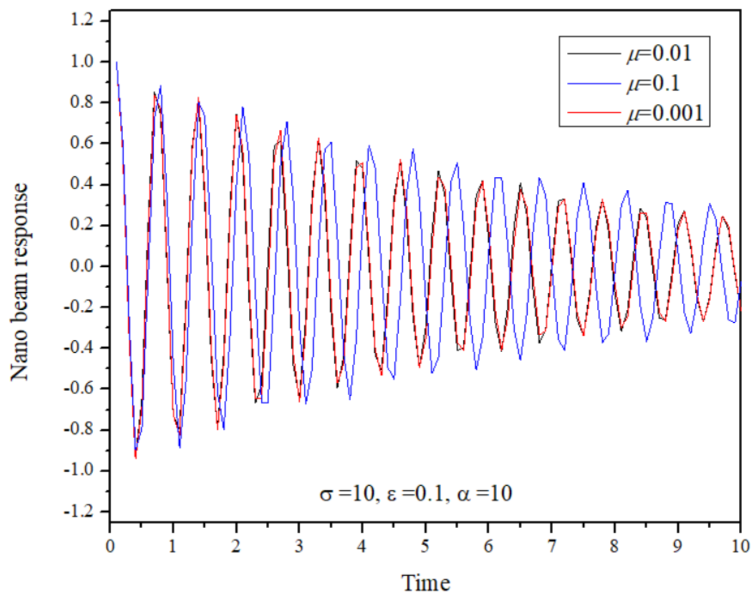

Fig. 6 Nanobeam response under the bound noise process

4. CONCLUSIONS

This paper considered a nanobeam axially compressed on its ends, where the compression forces were simulated as real noise and bound noise processes. First, a theoretical description of typical broadband stochastic processes (white noise, real noise, and bound noise) was given. Each one was analytically analyzed and the basic equations that characterize them were presented. Then the simulation of the real noise and bound process using MATLAB software was performed. Dynamic system behavior of a simple axially compressed nanobeam under the influence of the already defined stochastic processes was graphically presented. The simulation of the real noise and bound noise processes was shown, and it contains process values in the time domain of 10s. At the end of the study, the nanobeam responses under the presented processes were analyzed in the function of the nonlocal parameter μ.

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