**INNOVATIVE MECHANICAL ENGINEERING** **ISSN 2812-9229 (Online)**
University of Niš, Faculty of Mechanical Engineering
VOL. 2, NO 1, 2023, PP. 86 - 94

**Original scientific paper** *

# DETERMINING FLOW VELOCITY THROUGH STRAIGHT PLANAR PROFILE CASCADES BY USING CONFORMAL MAPPING ONTO BAND

**Saša Milanović[1], Miloš Jovanović[1], Jasmina Bogdanović Jovanović[1], Živan Spasić[1], Boban Nikolić[1]**

[1]University of Niš, Faculty of Mechanical Engineering in Niš, Serbia

**Abstract**. *This paper analyzes the mapping of flow nature around the profile of a straight plane cascade onto band with symmetrically distributed singular points depending on the geometric parameters of the cascade. In accordance with the character of variation of the velocity potential along the band contour, one can conclude that the whole contour is mapped onto a finite part of the band, so that the infinite reach of the band and the decaying conformity of mapping in infinity cannot cause difficulties while solving the problem. The Schwartz-integrals forming the mathematical model can be reduced to the forms with finite boundaries.*

**Key words**: *Profile, Flow, Straight planar cascade, Conformal mapping*

## 1. Introduction

Flow in turbomachinery is spatial, non-stationary and turbulent, and the working fluid is compressible and viscous. These are reasons for the usual simplifications.. The Reynolds number which characterizes the flow through turbomachinery blade systems is relatively large, which means that the boundary layer around the profile can be neglected and that it can be treated as a non-viscous fluid flows through the entire flow space. Another simplification is related to neglecting non-stationary flow in the turbomachinery bodies and reducing the real spatial flow in turbomachinery onto the two-dimensional flow.

This two-dimensional potential flow is solved using the conformal mapping method [1]. Flow through straight cascades is conformally mapped onto a certain simpler flow in an auxiliary plane and used in solving the direct problem [2]. The direct problem of the theory of flow through straight planar profile cascades is used to describe the flow through a given profile cascade, for a given front stagnant point on the profile and a given flow velocity intensity far in front of the profile cascade. Flow through a straight planar

***Received: January 12, 2023 / Accepted: May 22, 2023.**
**Corresponding author**:
University of Niš, Faculty of Mechanical Engineering
E-mail: sasa.milanovic@masfak.ni.ac.rs

cascade can be mapped onto flow through circle or plate cascade and flow around simple contours in a multilayer auxiliary plane (mapping onto a canonical region). The most suitable canonical regions for mapping are unit circles and bands [3, 4].

The analyzed mapping character shows that the infinite stretching of a band and the compromised conformity of mapping to infinity do not cause difficulties in solving the problems, and that it is not necessary to close the band with other contours [5, 6].

## 2. FLOW MAPPING CHARACTERISTICS

Of utmost importance is to determine the distribution of velocity per profile contour and direction of the outlet from the profile cascade, in various cascade fluid flow inlets. When an incompressible fluid flows through straight planar profile cascades, these kinematic flow characteristics are calculated using the potential flow model, while such problems can also be solved using the conformal mapping method [7-9].

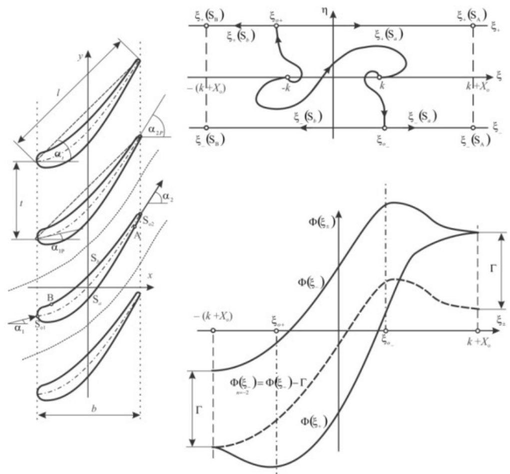


**Fig.1** Nature of mapping of profile contour into band contour

When solving a problem using the conformal mapping method, an infinite band $-\pi/2 \le \operatorname{Im} Z \le \pi/2$ is selected as a canonical region of flow mapping around the cascade profile with symmetrically distributed singular points in $Z = \pm k$, where $k$ is the real number depending on the geometric parameters of the cascade. Both infinitely removed regions in front of and behind the profile cascade are mapped. Flow in a band is an image of a flow between congruent streamlines at a cascade step distance; profile sides are mapped onto band contours $\xi_+(Z=\xi+i\,\pi/2)$ and $\xi_-(Z=\xi-i\,\pi/2)$ (Fig. 1).

The value of the velocity potential function along the band contour, with the accuracy up to the additive constant, is defined by the following expression:

$$\Phi_o(\xi_\pm) = \frac{Q}{2\pi}\left[\ln\frac{ch(\xi_\pm + k)}{ch(\xi_\pm - k)} \pm tg\alpha_1 arcctgsh(\xi_\pm + k) \mp tg\alpha_2 arcctgsh(\xi_\pm - k)\right] \quad (1)$$

where the values of function *arcctg* are placed into the interval (0÷π). Due to the ambiguity of function *arcctg*, the other values of velocity potential function along the band contour are:

$$\Phi_n\left(\xi_\pm\right) = \Phi_o\left(\xi_\pm\right) \pm Q\frac{tg\alpha_1 - tg\alpha_2}{2}n \quad , \quad n = o, \pm 1, \pm 2, \ldots \tag{2}$$

where $Q$ is the cascade step flow ($Q = v_x \cdot t$), and $\alpha_1$ and $\alpha_2$ are the angles of flow directions in front of and behind the cascade (Fig. 1).

Velocity distribution along the band contour $V\left(\xi_\pm\right) = d\Phi\left(\xi_\pm\right)/d\xi_\pm$ is given as:

$$V\left(\xi_\pm\right) = \frac{Q}{2\pi}\frac{\mp tg\alpha_1 \cdot ch\left(\xi_\pm - k\right) \pm tg\alpha_2 \cdot ch\left(\xi_\pm + k\right) + sh2k}{ch\left(\xi_\pm + k\right)ch\left(\xi_\pm - k\right)} \tag{3}$$

Stagnant points in band contours, in which stagnant points at the front ($S_{o1}$) and the rear ($S_{o2}$) of the cascade profile are mapped, are determined using the following expression:

$$\xi_{o+} = k + \ln\frac{sh2k\left(-1 \pm \sqrt{1+A}\right)}{B} \quad , \quad \xi_{o-} = k + \ln\frac{sh2k\left(1 \pm \sqrt{1+A}\right)}{B} \tag{4}$$

where A and B are the constants with values:

$$A = \frac{2tg\alpha_1 \cdot tg\alpha_2}{th2k \cdot sh2k} - \frac{tg^2\alpha_1 + tg^2\alpha_2}{sh^2 2k} \quad , \quad B = e^{2k}tg\alpha_2 - tg\alpha_1$$

Fig. 1 shows the mapping character (when the front stagnant point $So_1$ is mapped onto $\xi_{o+}$, and the rear stagnant point $S_{o2}$ onto $\xi_{o-}$),which is the most common case of flow in impactless inlet to profile cascades in which inclination angles of the profile skeleton line at the cascade inlet and outlet are greater than zero ($\alpha_{1P}>0$ and $\alpha_{2P}>0$). According to the labels in Fig. 1, the stagnant point on the profile inlet edge ($S_{o1}=0$) is mapped onto the stagnant point on the upper band contour ($\xi_{o+}$), while the stagnant point on the outlet edge ($S_{o2}=0$) is mapped onto the stagnant point on the lower band contour ($\xi_{o-}$). The profile side marked with $S_a$ is partially mapped onto the upper band contour $\xi_{o+} \le \xi_+ \le \infty$, only for the mapping to continue on the lower band contour $\xi_{o-} \le \xi_- \le \infty$. The opposite profile side, marked with $S_b$, is mapped onto $-\infty \le \xi_+ \le \xi_{o+}$ and $-\infty \le \xi_- \le \xi_{o-}$.

Analysis of the change in the velocity potential character along the band contour $\Phi(\xi_\pm)$ (2) shows that it quickly tends to asymptotic boundary values and that already at distances $\xi_\pm = -(k + X_o)$ and $\xi_\pm = k + X_o$ it reaches the range of the adapted values of a negligible change in the velocity potential $\varepsilon_\Phi$, for:

$$X_o = arcshctg\frac{\pi \cdot \varepsilon_\Phi}{Q \cdot \left|tg\alpha_{1,2}\right|} \tag{5}$$

where $\alpha_{1,2}$ are larger than angles $\alpha_1$ or $\alpha_2$ in absolute value. This physically means that on the parts of the band contour$\xi_\pm \geq k + X_o$ and $\xi_\pm \leq -(k + X_o)$ such small parts of the profile contour are mapped that they can be considered "*points*" in the calculations. According to the mapping shown in Fig. 1, "point" $S_A$ of the profile contour is mapped with infinite stretching onto contours $\xi_\pm \geq k + X_o$, while "*point*" $S_B$ of the profile contour is mapped onto $\xi_\pm \leq -(k + X_o)$.

In impactless inlet to the cascades in which the profile skeleton line inclination angles at the entrance of the cascade ($\alpha_{1P}$) are close to zero or less than zero and in which the profile skeleton line inclination angles at the exit of the cascade ($\alpha_{2P}$) are close to zero or less than zero, the front and the rear stagnant points are mapped on the same side of the band. Regardless of the mapping spot of the stagnant points, the mapping character is such that the entire profile contour is mapped onto the limited part of the band $-(k + X_o) \leq \xi_\pm \leq k + X_o$, while only two "*points*" of the profile contour are mapped with infinite stretching onto $\xi_\pm \leq -(k + X_o)$ and $\xi_\pm \geq k + X_o$.

## 3. Problem-Solving Equation

According to Zhukovsky-Chaplygin the rear stagnant point on the profile ($S_{o2}$) is placed on the profile outlet edge. For a fixed position of the rear stagnant point, this is enough to determine by conformal mapping only one (it does not matter which one) flow through the given profile cascade. As a result of this solution one can obtain: the canonical region parameter (k) and the law for correspondence between points on the profile and the band contour ($\xi_\pm$ (s)); the determination of other flows is very simple [7].

Since the velocity direction inclination angle is known along the profile contour $\alpha(s)$, and as it is equal in potential flow to the inclination angles of the profile contour tangents, determining the velocity distribution along the profile contour ($v(s)$) and the streamlines inclination angles in front of ($\alpha_1$) and behind ($\alpha_2$) the cascade is solved as a contour task in the canonical region for the analytical function: $F(z) = i \ln \bar{v} = \alpha + i \ln v$, mapped in the plane of the selected canonical region. The Schwartz integrals for determining the real parts of the considered analytical function in $Z = -k$ (where $\alpha = \alpha_1$) and $Z = k$ (where $\alpha = \alpha_2$) define the angles $\alpha_1$, $\alpha_2$:

$$\alpha_1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\alpha_+(t) + \alpha_-(t)}{ch(t+k)} dt \qquad , \qquad \alpha_2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\alpha_+(t) + \alpha_-(t)}{ch(t-k)} dt \tag{6}$$

$$\ln v(\xi_\pm) = \frac{1}{2\pi}\left[ \pm \int_{-\infty}^{\infty} \alpha_\pm cth \frac{t-\xi}{2} dt \mp \int_{-\infty}^{\infty} \alpha_\mp th \frac{t-\xi}{2} dt - \int_{-\infty}^{\infty} (\alpha_+ - \alpha_-) tht dt \right] + \ln v(0) \tag{7}$$

Constant $\ln v(0)$ in expression (7) is determined by using the expression:

$$\ln v(0) = \frac{1}{2}\ln(v_1 \cdot v_2) + \frac{shk}{2\pi}\int_{-\infty}^{\infty} (\alpha_+ - \alpha_-) \frac{tht}{ch(t+k)(t-k)} dt \tag{8}$$

In the case when the front stagnant point is mapped onto the upper band contour, and the rear point is mapped onto the lower contour, the Schwartz-integrals (6, 7) for determining angles $\alpha_1$, $\alpha_2$ and velocity functions along the profile contour mapped onto the band are reduced to:

$$\alpha_1 = \frac{1}{2\pi}\int_{-(k+Xo)}^{k+Xo}\frac{\alpha_+ + \alpha_-}{ch(t+k)}dt + \frac{2\alpha_B}{\pi}arctge^{-Xo} + \frac{2\alpha_A}{\pi}\left(\frac{\pi}{2} - arctge^{(Xo+2k)}\right)$$

$$\alpha_2 = \frac{1}{2\pi}\int_{-(k+Xo)}^{k+Xo}\frac{\alpha_+ + \alpha_-}{ch(t-k)}dt + \frac{2\alpha_B}{\pi}arctge^{-(Xo+2k)} + \frac{2\alpha_A}{\pi}\left(\frac{\pi}{2} - arctge^{Xo}\right) \tag{9}$$

$$\ln v(\xi_\pm) = I(\xi_\pm) - C + \ln v(0) \tag{10}$$

where:

$$I(\xi_\pm) = \frac{1}{2\pi}\left[\pm\int_{-(k+Xo)}^{k+Xo}\alpha_\pm cth\frac{t-\xi_\pm}{2}dt \mp \int_{-(k+Xo)}^{k+Xo}\alpha_\mp th\frac{t-\xi_\pm}{2}dt \pm 2\alpha_B\, ln\, t\, h\frac{k+X_o+\xi_\pm}{2} \mp 2\alpha_A\, ln\, t\, h\frac{k+X_o-\xi_\pm}{2}\right]$$

$$C = \frac{1}{2\pi}\int_{-(k+Xo)}^{k+Xo}(\alpha_+ - \alpha_-)\, thtdt \qquad \text{and}$$

$$ln\, v\,(0) = \frac{1}{2}ln(v_1\cdot v_2) + \frac{shk}{2\pi}\int_{-(k+Xo)}^{k+Xo}(\alpha_+ - \alpha_-)\frac{tht}{ch(t+k)ch(t-k)}dt$$

Since the mapping function $\xi_\pm(s)$ and the canonical region parameter $k$ (which determines the position of singular points) are also unknown in addition to $\alpha_1$, $\alpha_2$ and $v(s)$, Schwartz integrals (6, 7) (i.e. 9, 10) are expanded with two more equations, in line with the condition of equality between the fluid flows along the corresponding contours:

$$\int_{So1=0}^{(So2)a} v(S_a)dsa = \Phi(\xi_{o2}) - \Phi(\xi_{o1}), \quad \xi_{o1} = \xi_\pm(S_{o1}), \xi_{o2} = \xi_\pm(S_{o2}) \tag{11}$$

and

$$\int_{So1}^{S} v(s)ds = \Phi_n[\xi_\pm(s)] - \Phi(\xi_{o1}), \text{ for } S = S_a \in [0,(S_{o2})_a] \text{ and } S = S_b \in [0,(S_{o2})_b] \tag{12}$$

which closes the system of equations needed to solve the problem. Equation (11) is used to determine the canonical region parameter ($k$), while equation (12) is used to determine the mapping function $\xi_\pm(s)$.

A flow with impact less inlet is solved as a basic task, with the front stagnant point on the profile inlet edge. For the given flow velocity component $v_x(v_x=1)$, $\alpha_1$, $\alpha_2$, $v(s)$, the correspondence $\xi_\pm(s)$ and parameter $k$ are determined. The task is solved through iteration. In the initial approximation it is assumed that $\alpha_1=\alpha_{1P}$ and $\alpha_2=\alpha_{2P}$ (where $\alpha_{1P}$ and $\alpha_{2P}$ are the profile skeleton line inclination angles on the inlet and the outlet edge, Fig. 1), assuming the velocity intensities do not change along the profile sides. According to the fluid flow intensity along the profile side $S_a$, by using equation (11) one can determine the canonical region parameter ($k$). Parameter $k$ is determined through iteration, which makes the solution of the task doubly iterative.

## 4. Problem-Solving Procedure

The procedure for solving a flow with impactless inlet along the straight planar profile cascade mapped onto a band is given in the algorithm in Fig. 2. As can be seen from the algorithm, the problem-solving procedure is iterative along angle $\alpha_1$. If a 0.1% solution convergence is achieved, the task is solved, if not, the iterative procedure continues.

Special attention is paid to determining the canonical region parameter ($k$). According to the fluid flow intensity along the profile side $S_a$, the canonical region parameter ($k$) is determined by using equation (11). Parameter $k$ is also determined through iteration thus the solution of the task is doubly iterative. In the initial approximation it is assumed that $k=k_o$, where $k_o$ is the value of the canonical region parameter in mapping the cascade of thin plates with density $l/t$ and inclination angle $\alpha_T$, equal to the inclination angle of the cascade profile skeleton line chords for which the task is being solved. In the task-solving program parameter $k$ is calculated with accuracy to the third decimal place ($\Delta k$=0.001).

Upon determining parameter $k$ and stagnant points on the band contour, according to equation (12) one can determine the mapping function $\xi_\pm(s)$ in the first approximation. Upon determining function $\xi_\pm(s)$, that is, $S(\xi_\pm)$, function $\alpha(\xi_\pm)$ is determined in line with the known function of the tangent inclination angle along the profile contour $\alpha(s)$, followed by the determination of angles $\alpha_1$ and $\alpha_2$ and velocity $v(\xi_\pm)$ using the Schwartz-integrals in the selected computational points $\xi_\pm(s)$; $v(\xi_\pm) = v\left[\xi_\pm(s)\right] = v(s)$.

By using the results for $\alpha_1$, $\alpha_2$ and $v(s)$ obtained in the first approximation, the task is solved in the second approximation and so on. The iterative loop stops when the results negligibly deviate from the results obtained in the previous iteration. The obtained results are verified by comparing them to the velocity distribution along the profiles obtained using the Schlichting, Fuzzy and PB method, singularity method, and conformal mapping onto a band with a singularity in zero [4]. The canonical region parameter ($k$) and mapping function $\xi_\pm(s)$ are obtained as the results of the calculation.

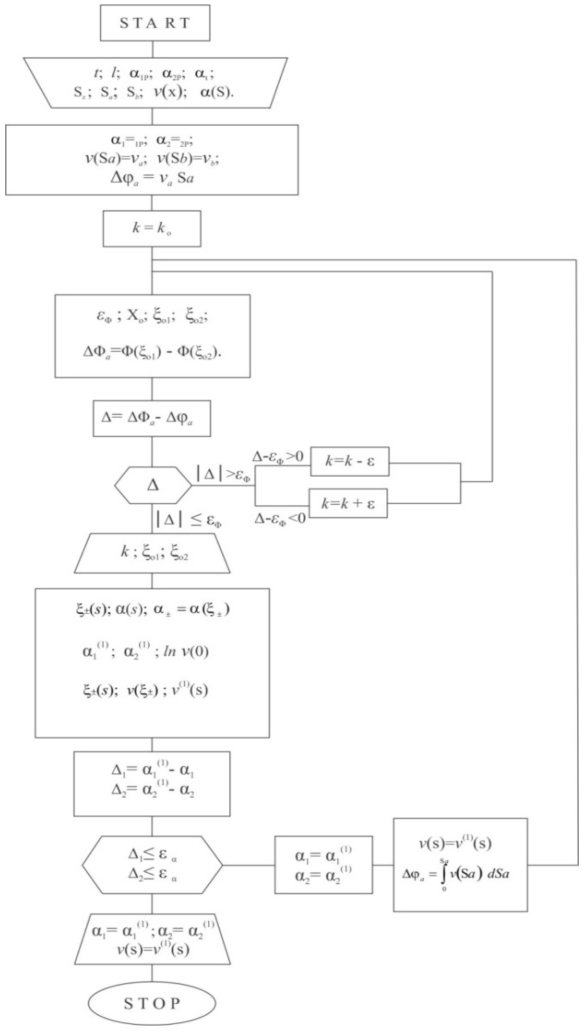
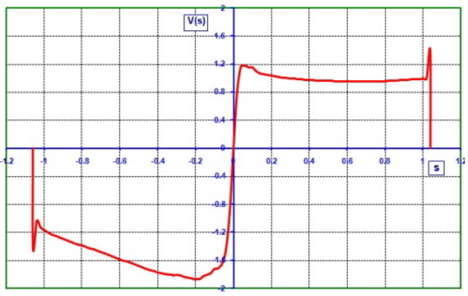


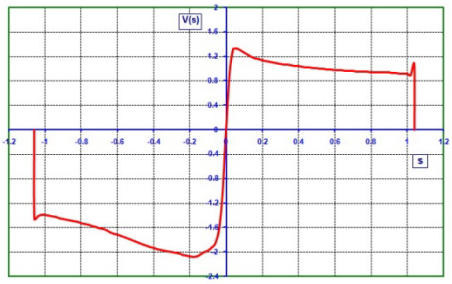

**Fig.2** Block diagram

## 5. Velocity Distribution Along The Profile

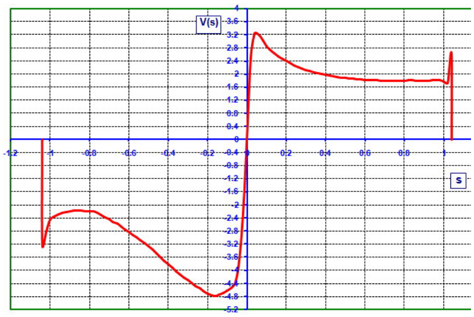
The following figures present the distributions of velocities along the NACA 8410 profile of a real planar cascade.



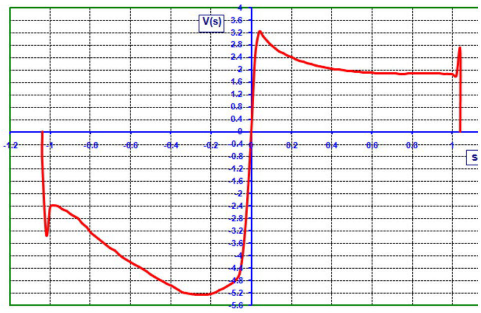

*a)* $\alpha_{1P}$=56°; $\alpha_{2P}$=16°; $\alpha_T$=56° $l/t$=0,67
$\alpha_1$=42,57°; $\alpha_{2P}$=25,1°; $k$=1,235




*b)* $\alpha_{1P}$=56°; $\alpha_{2P}$=16°; $\alpha_T$=56° $l/t$=1
$\alpha_1$=45,28°; $\alpha_{2P}$=22,8°; $k$=1,844




*c)* $\alpha_{1P}$=76°; $\alpha_{2P}$=46°; $\alpha_T$=60° $l/t$=1,5
$\alpha_1$=79,67°; $\alpha_{2P}$=55,05°; $k$=3,95




*d)* $\alpha_{1P}$=76°; $\alpha_{2P}$=46°; $\alpha_T$=60° $l/t$=1
$\alpha_1$=72,33°; $\alpha_{2P}$=56,38°; $k$=3,028

**Fig. 3** Velocity distribution along the profile

## 6. Conclusion

According to the character of flow mapping, one can conclude that the infinite stretching of a band does not lead to additional difficulties in solving the direct problem. Closing the band with other contours, e.g. unit circles, as a way of solving this problem is unnecessary, and could lead to further complications. In line with the numerically solved examples for various profile cascades, it can be confirmed that the band, as a flow mapping region, enables the solution of the direct problem for all types of profile cascades. The computational program is set such that it can be used for cascades of various geometries, by inputting the minimum of parameters. In the iterative procedure, a criterion for verifying the accuracy of the obtained results is established in the form of matching angles $\alpha_1$ from two adjacent iterations.

**Acknowledgement:** The research was financially supported by the Republic of Serbia Ministry of Education, Science and Technological Development (Contract No. 451-03-68/2022-14/ 200109)

## REFERENCES

1. Ghosh N., Karmakar T., Raja Sekhar G. P., Application of conformal mapping to two-dimensional flows in an anisotropic aquifer, DOI: 10.1007/s13226-021-00153-1, Indian Journal of Pure and Applied Mathematics, 2021.
2. Etinberg I.E., Rauhman B.S., Гидродинамика гидротурбин, Lenjingrad 1978.
3. Stepanov G.O., Гидродинамическая сетка турбомашины, Moskva, 1962.
4. Bogdanović B., Određivanje rasporeda brzine po konturi profila prave rešetke preslikavanjem strujanja na pojas, doktorska disertacija, Mašinski fakultet u Nišu, Niš 1982.
5. Bogdanović B., Milanović S., Solution of the direct problem in theory of flow through straight plane profile cascade by using conformal mapping into band $-\pi/2 \le \operatorname{Im} Z \le \pi/2$, Facta Universitatis, series: Mechanical Engineering, Vol. 1, No7, (2000), pp. 809-816,
6. Milanović S., Proračun prostornog strujanja kroz aksijalne turbomašine kao kompleks dva dvodimenzijska strujanja, magistarski rad, Mašinski fakultet u Nišu, Niš 1996.
7. Bogdanović B., Proračun potencijalnog strujanja kroz prave ravanske rešetke profila metodom konformnog preslikavanja, monografija Mašinski fakultet u Nišu, Niš 1999.
8. Deič M. E., Samojlovič G.S. Основы аэродинамики осей турбомашин, Moskva 1959.
9. Loicijanski L. G., Механика жидкостей и газов, Moskva 2003.