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NONLINEAR DYNAMICS OF BALL ROLLING OF RADIAL BEARINGS IN AN ASSEMBLY VITH AN UNBALANCED MULTI-STAGE ROTOR

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Abstract.*The paper presents a mathematical analysis of the kinematics and dynamics of rolling balls of an ideal radial ball bearing assembly of eight or twelve balls, between the stationary circular groove and the movable one mounted on the shaft, which rotates at the appropriate angular velocity. Ordinary non-linear differential equations and equations of phase trajectories of the assembly were derived for two cases: a) when the eccentricity of the center of mass of one or more pairs of radial bearing balls occurs in the rolling bearing and when the rotor is balanced; and b) when the non-linear dynamics of the rolling of the ball bearings occurs in the rolling bearing caused by the non-linear dynamics of an unbalanced, single-stage or multi-stage rotor.*

Key words:*Nonlinear dynamics, Rolling ball, Radial ball bearings, Unbalanced rotor, Unbalanced radial ball bearings, Multi-stage rotor*

1. INTRODUCTION

In the last five or six years, the author has published a series of papers (among them References [7-11]) with new results on the nonlinear dynamics of the rolling, no-slip, heavy ball on curvilinear trajectories and surfaces of different shapes. Some of these published papers also contain the author's new analytical theory of the collision of rolling bodies (see Ref. [11], and see also two first doctorates [1, 2] of rolling balls, defended in 1923 and 1932).

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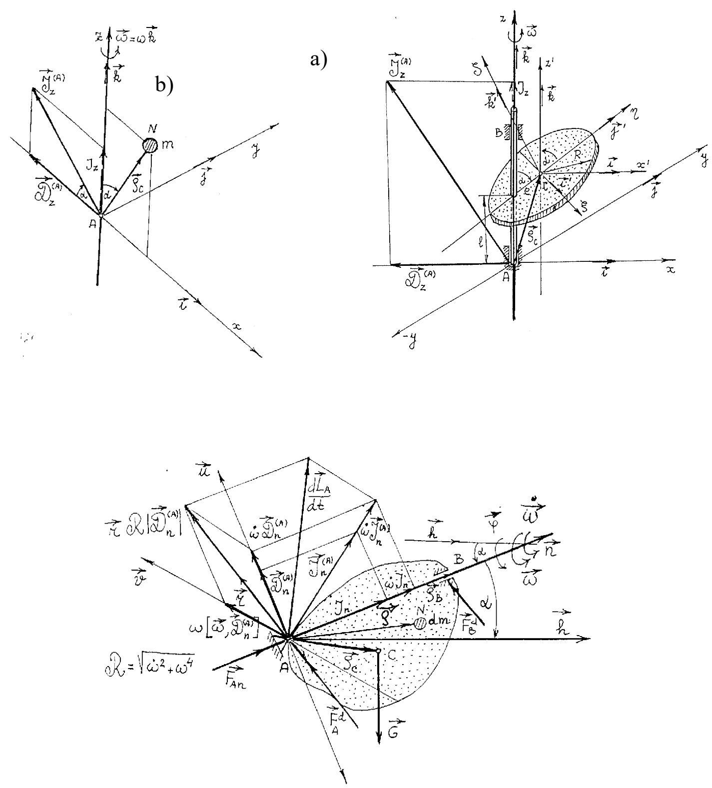
At the 1992 IUTAM Congress ICTAM Haifa, the author presented the mass moment vectors, $\vec{\mathsf{J}}_{\vec{n}}^{(O)}$, the mass inertiamoment vector and the static linear mass moment vector, $\vec{\mathsf{S}}_{\vec{n}}^{(O)}$, related to the pole O and the axis oriented by a unit vector $\vec{n}$ in the form (for details see References [3, 4, 5, 6] and Figure 1.a and b):

$$\vec{\mathsf{J}}_{\vec{n}}^{(O)} \overset{def}{=} \iiint_V [\vec{\rho},[\vec{n},\vec{\rho}]]dm \text{ and } \vec{\mathsf{S}}_{\vec{n}}^{(O)} \overset{def}{=} \iiint_V [\vec{n},\vec{\rho}]dm \quad (1)$$

In the previous expressions, $\vec{\rho}$ the position vector of the elemental mass dm of the body V is the volume of the body in which the body is placed.

In Fig. 1.a), the graphical presentation of the vector of mass particle's mass inertia moment for the reference point and an oriented axis and of the corresponding deviational plane is presented.



c)

Fig. 1 a) The graphical presentation of the vector of mass particle's mass inertia moment for the reference point and an oriented axis and of the corresponding deviational plane; б)

The graphical presentation of the vector of mass inertia moment for the reference point and an oriented axis and of the corresponding deviational plane with axial and deviational mass inertia components, of eccentrically positioned disk on the shaft; c) The graphical presentation of the vector of mass inertia moment for the reference point in stationary bearing and an oriented axis if shaft of unbalanced rotor, and kinetic pressure and reaction shaft bearings;

In Fig.1.b, the graphical presentation of the vector of mass inertia moment for the reference point and an oriented axis and of the corresponding deviational plane with axial and deviational mass inertia components, of eccentrically positioned disk on the shaft are presented.In Fig. 1.c, the graphical presentation of the vector of mass inertia moment for the reference point in stationary bearing and an oriented axis if shaft of unbalanced rotor, and kinetic pressure and reaction shaft bearings are presented.

The theorem in mathematical form, for the mass inertia moment vector $\vec{J}_{\vec{n}}^{(O)}$, related to the pole $O \equiv A$ and the axis oriented by the unit vector $\vec{n}$ is valid following expression:

$$\vec{J}_{\vec{n}}^{(O)} = \vec{J}_{\vec{n}}^{(C)} + [\vec{\rho}_C, [\vec{n}, \vec{\rho}_C]]M \qquad (2)$$

in which C is the center of mass of the body, $\vec{\rho}_C$ the vector of the position of the pole $O \equiv A$ in relation to the center of mass C of the body, M is the total mass of the body. $\vec{J}_{\vec{n}}^{(C)}$ is the mass inertia moment vector for the pole at the center C of mass of the body and for the parallel axis through the center of mass C oriented by the same unit vector $\vec{n}$. The term $[\vec{\rho}_C, [\vec{n}, \vec{\rho}_C]]M$ in expression (2) represents the positional moment of inertia of the mass of the body when the axis passes through the pole O in relation to the parallel axis through the center of mass C.

Also, a new rotator vector $R = \sqrt{\dot{\omega}^2 + \omega^4}$ was introduced, purely kinematic, which depends only on the angular velocity and angular acceleration of the shaft rotation. Using this rotator vector R , the analysis showed that the kinetic pressures on the shaft bearings do not rotate at the same angular velocity as the shaft, if the shaft also has an angular acceleration. Kinetic pressures on the shaft bearings are expressed by means of the vector of the deviation moment $\vec{D}_{\vec{n}}^{(A)}$ of the mass of the body, which is related to the pole O and the axis oriented by the unit vector $\vec{n}$ and is the orthogonal component of the mass immersion moment vector $\vec{J}_{\vec{n}}^{(O)}$ related to the pole O and the axis oriented by the unit vector $\vec{n}$, while the axial component is in the direction of the axis itself.

The vector equations of the dynamic balance-rotation of the shaft with an eccentrically placed disk on the shaft (see Figure 1.b) are written using the change $\frac{d\vec{K}}{dt}$.in the linear momentum (impulse) and the change $\frac{d\vec{L}_A}{dt}$ in the angular momentum (momentum of the impulse) of body rotation movement in in the following form:

$$\frac{d\vec{K}}{dt} = \left|\vec{S}_{\vec{n}}^{(A)}\right|\left(\dot{\omega}\vec{u}_1 + \omega^2 \vec{v}_1\right) = \vec{R}\left|\vec{S}_{\vec{n}}^{(A)}\right| = R\left|\vec{S}_{\vec{n}}^{(A)}\right|\vec{r}_1 = \sum_{k=1}^{k=N}\vec{F}_k + \vec{F}_A + \vec{F}_B ,$$

$$\frac{d\vec{\mathsf{L}}_A}{dt} = \dot{\vec{\omega}} J_{\vec{n}}^{(A)} + \dot{\omega}\vec{\mathsf{D}}_{\vec{n}}^{(A)} + \omega\left[\vec{\omega}, \vec{\mathsf{D}}_{\vec{n}}^{(A)}\right] = \dot{\vec{\omega}} J_{\vec{n}}^{(A)} + \left|\vec{\mathsf{D}}_{\vec{n}}^{(A)}\right| \vec{\mathsf{R}} = \sum_{k=1}^{k=N} \left[\vec{\rho}_k, \vec{F}_k\right] + \left[\vec{\rho}_B, \vec{F}_B\right] \tag{3}$$

From the previous system of vector equations (3) of the dynamic balance of the rotor, we obtain the reactions of the shafts in the forms:

$$\vec{F}_{A\vec{n}} = \left(\vec{F}_A, \vec{n}\right)\vec{n} = -\vec{n}\sum_{k=1}^{k=N}\left(\vec{F}_k, \vec{n}\right) - \vec{n}\left(\vec{G}, \vec{n}\right),$$

$$\vec{F}_{AT} = -\vec{F}_B + \vec{\mathsf{R}}_1\left|\vec{\mathsf{S}}_{\vec{n}}^{(A)}\right| - \left[\vec{n}, \left[\vec{G}, \vec{n}\right]\right] - \sum_{k=1}^{k=N}\left[\vec{n}, \left[\vec{F}_k, \vec{n}\right]\right],$$

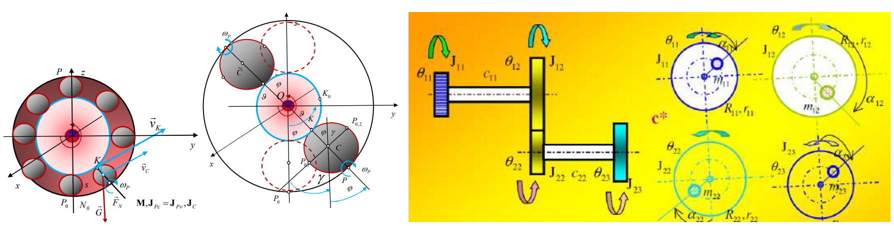
$$\vec{F}_B = \frac{1}{\rho_B}\vec{\mathsf{R}}\left|\vec{\mathsf{D}}_{\vec{n}}^{(A)}\right| - \frac{1}{\rho_B}\left[\vec{n}, \left[\left[\vec{\rho}_C, \vec{G}\right], \vec{n}\right]\right] - \frac{1}{\rho_B}\sum_{k=1}^{k=N}\left[\vec{n}, \left[\left[\vec{\rho}_k, \vec{F}_k\right], \vec{n}\right]\right], \tag{4}$$

as well as the shaft rotation equation in the form:

$$\left(\vec{\mathsf{J}}_{\vec{n}_i}^{(A_i)}, \overset{*}{\vec{\omega}}_i\right) - \left(\vec{n}_i, \left[\vec{\rho}_{C_i}, \vec{\mathbf{G}}_i\right]\right) = \left(R_{i1}\mathbf{F}_{i1} - R_{i2}\mathbf{F}_{i2}\right) \tag{5}$$

in the function of active forces and deviation properties of the rotor. In the expression (4b), $\vec{R}_1\left|\vec{S}_{\vec{n}}^{(A)}\right|$ and in expression (4c) $\frac{1}{\rho_B}\vec{R}\left|\vec{D}_{\vec{n}}^{(A)}\right|$ the terms represent the kinetic forces pressures on the shaft bearings, while ρ_B is the distance between the shaft bearings.

The previous results were an inspiration for the application of the obtained results to the nonlinear dynamics of rolling balls in radial shaft bearings of balanced and unbalanced rotors.



a)b) c)

Fig. 2 Elements of a radial ball bearing system and a two-stage rotor: a) schematic representation of a radial ball bearing with eight balls; b) representation of one of the pairs of balls on one diameter of the concentric circular contours in the radial bearing, with indicated arcs of contact of one of the balls, in rolling, with them; and c) a sketch of the model of the two-stage rotor with discs, the model of its reduction to a single shaft and the scheme of the discs with imbalances and indicated angular velocities

In Fig. 2, the graphical presentation of the elements of a radial ball bearing system and a two-stage rotor are presented as well as the content of this paper. In Fig. 2.a), the schematic representation of a radial ball bearing, with eight balls is visible. In Fig. 2.b), the representation of one of the pairs of balls on one diameter of the concentric circular contours in the radial bearing, with indicated arcs of contact of one of the balls, in rolling, with them is presented.In Fig. 2.c), a sketch of the model of the two-stage rotor with discs, the model of its reduction to a single shaft and the scheme of the discs with imbalances and indicated angular velocities are presented.

2. DESCRIPTION OF THE ROTOR MODEL WITH RADIAL BALL BEARINGS

Figure 3 shows a model of a two-stage power transmission and rotation transmission with two shafts and two disks (gears) on each shaft, and each shaft is supported on two radial ball bearings, whose middle planes are vertical. The same Figure 3 shows two models of radial ball bearings with four and six pairs of two balls on each diameter of the radial ball bearings. Two balls, on one diameter, represent one pair in a radial ball bearing. Since all the balls in one radial ball bearing have the same radius r and if their mass densities are equal, then the center of mass of the radial ball bearing is in the center of the concentric circular paths, the fixed circular groove, the radius R of which the balls roll, without slipping.

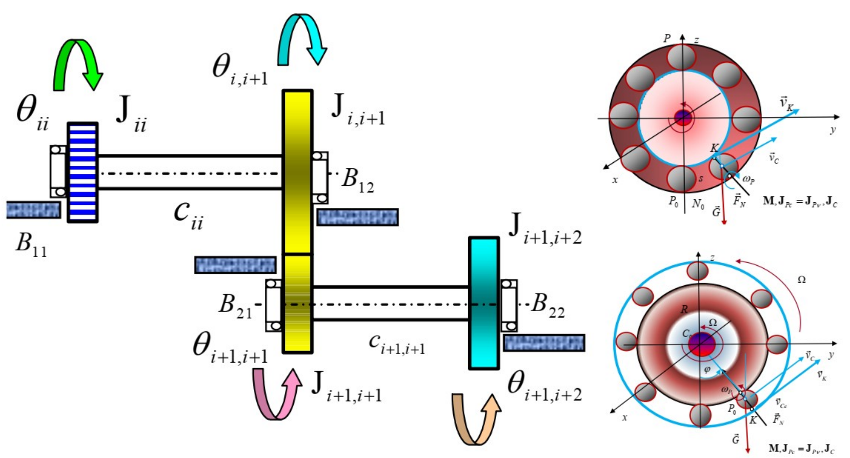


Fig. 3 Configuration of radial ball bearings on the shafts of a two-stage gear transmission with unbalanced gears (with debalances in the form of eccentrically placed material points).

The balls in the rolling, without sliding, are in permanent contact, without sliding, with a movable circular groove, with a radius of $R-2r$. They are constantly and permanent at the same distance from each other and in rolling retain the initial relative configuration. Models of radial ball bearings are simplified by neglecting oil lubrication, and that there is no chain "net", as in real radial ball bearings, which maintains the distances between them, which is explained by assuming that the balls roll without sliding along the stationary circular groove, and that they are made by kinematic contact, without sliding, along the movable circular groove of the radial ball bearing, fixed on corresponding shaft. In the ideal case, a balanced rotor - shaft and gears, and all pairs of balls of radial ball bearings are balanced, the system rotates at a constant angular velocity, there are no kinetic pressures on the balls of radial ball bearings, except for reaction forces due to the weight of the shaft and gears carried by the shaft of the action of external active forces.

In this paper, we will present the results of the investigation, individually, of the unbalance of gear shafts and especially the unbalance of radial ball bearings, due to the eccentricity of the center of mass of a pair of balls on one of the diameters, or several of them.

We will determine the nonlinear differential equations for each of the cases, as well as the equations of phase trajectories and sketch the phase portraits of the nonlinear dynamics of the rotation of the shaft, as well as the nonlinear dynamics of rolling balls in radial spherical bearings. We will also study the nonlinear dynamics of shaft rotation, to which we have reduced the shafts and unbalanced gears and the shafts of a multi-stage rotation transmission (with multiple number of shafts and unbalanced gears on them) and write the corresponding nonlinear differential equations and equations of phase trajectories.

2.1. Radial ball bearing models and ball rolling angular velocities

Fig.4 shows two variants of two types of radial ball bearings with four, Fig. 4a) and b), or six pairs of balls, Fig. 4c) and d), with two balls in a pair on each of the diameters of the radial ball bearing. One type of radial ball bearing is with balls rolling on the outer, Fig. 4a) (the other on the inner, Fig. 4b) stationary circular groove and in contact with the inner, Fig. 4c) (the other in contact with the outer, Fig. 4d) by a movable circular groove, which is fixed on the shaft, and rotates together with it at an angular velocity of $\dot{\vartheta}(t)$. The configuration of the balls in the radial ball bearing is determined by the central angleφ, based on the assumption that the rolling of the balls on a non-moving circular groove is non-slip, as well as that there is no sliding of contact between the balls and the moving circular groove, the position of the balls is determined by the angle φ and number of balls, as well as in the order of balls in a radial ball bearing. For each of the pairs is:

$$\varphi_k = \varphi + \frac{(k-1)\pi}{4} \ , \ \varphi_k = \varphi + \pi + \frac{(k-1)\pi}{4}, k = 1,2,3,4 \qquad (6)$$

and

$$\varphi_k = \varphi + \frac{(k-1)\pi}{6}, \quad \varphi_k = \varphi + \pi + \frac{(k-1)\pi}{6}, \ k = 1,2,3,4,5,6 \qquad (7)$$

In our research, we assume that the midplanes of the radial ball bearings are in vertical planes.

The angular rolling velocity of all balls and in each radial ball bearing, regardless of the number of balls, under the pre-assumed assumptions that there is no slipping both in rolling and in contacts, are equal to each other.

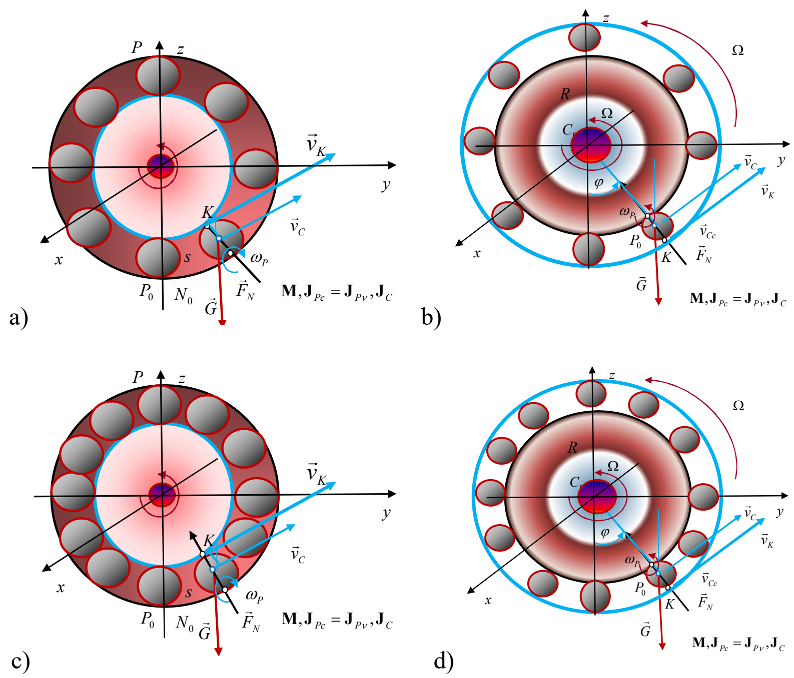


Fig. 4 Dynamic configuration of pairs of balls in radial ball bearings in four models of radial ball bearings with pairs of balls on one diameter: models with: a and b radial bearing with four pairs of two balls on each diameter and with a total of eight balls, in rolling without sliding; and c and d, a radial ball bearing with six pairs of balls, two balls on each diameter and with a total of twelve balls in rolling, without sliding, externally or internally in a fixed circular path, i.e. in dynamical contact with a moving circular path, internally or outside, fixed to the shaft in rotation.

2.2. Angular rolling velocities of balls in radial ball bearings

Let's adopt the central angle φ of the independent generalized coordinate of the dynamics of rolling one ball along a fixed circular circle. Denote by $\vartheta = \Omega t$ or $\vartheta(t)$ the angle of rotation of the shaft, to which the movable circular groove is rigidly connected, which is in kinematic contact with the ball, on which it rolls, without slipping, by moving the contact point along the movable circular groove.

For the first two models, Fig. 4a) and c), ideally the roller bearing assembly, we assume that the center of the cross section of shaft is returned to the center of the roller bearing and does not move, and that the shaft only rotates at a constant angular velocity $\dot{\vartheta} = \Omega = const$ or with variable angular velocity $\dot{\vartheta}(t) \neq const$ as a function of time. For other cases, we can assume that the center of the cross section of the shaft and the roller bearing moves, i.e., oscillates in the vertical direction, or that there are oscillations in two orthogonal directions, i.e., moving in a circular or elliptical path.

The instantaneous angular velocity ω_P of the ball rolling on a fixed circular groove and is the result of rolling, both due to change by rolling, without sliding, on a fixed circular groove and kinematic contact with a movable circular groove when pulling, or pushed due to rolling caused by kinematic rotation rotates at a constant angular velocity $\dot{\vartheta} = \Omega = const$ or variable $\dot{\vartheta}(t) \neq const$ together with the shaft and forces the ball to roll. Thus obtained, these two instantaneous angular rolling velocities arecoupled, because the system with one degree of rolling freedom has both central angles φandϑ. The independent generalized angular coordinate φ described on the surface of a fixed circular groove and the dependent angular coordinate ϑ described on the surface of a movable circular groove and therefore their dependence on each other should be found. And since we choose the angle coordinate φ as an independent generalized coordinate, the angular coordinate ϑ should be expressed over φ.

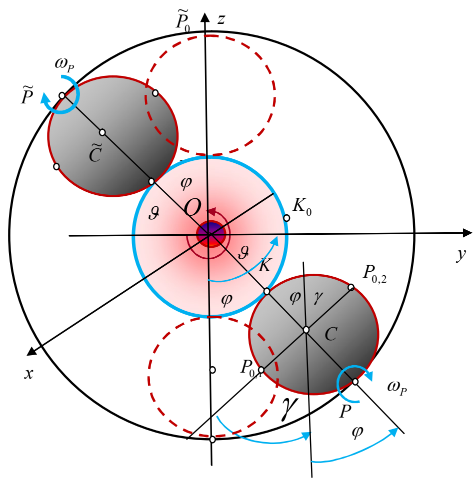


Fig.5 On the dynamics of rolling a pair of balls on a single diameter of a radial ball bearing, and on the appearance of nonlinear dynamics - characteristic geometric elements, characteristic points and arcs in correlation, non-slip, on fixed circular path and contact, non-slip, on circular contour of movable circular platform with shaft at a shaft's angular velocity.

Using the velocity v_C of the center of mass of the ball in rolling, we can easily determine the current angular velocity ω_P of a rolling the ball in the following form:

$$v_C = (R-r)\dot{\varphi} = r\omega_P \qquad \text{followed} \quad \omega_P = \frac{(R-r)\dot{\varphi}}{r} \tag{8}$$

Now we are using the kinematic connection of the contact of the ball and the movable circular groove, at the point K of contact, by determining its velocity v_K, as a point K, which belongs to the movable circular groove, which rotates at an angular angle velocity $\dot{\vartheta} = \Omega$, but also belongs to the ball contact point K in rolling at a current angular velocity $\omega_P = \frac{(R-r)\dot{\varphi}}{r}$ of ball rolling in the form:

$$v_K = (R-2r)\dot{\vartheta} = 2r\omega_P \text{followed} \quad \omega_P = \frac{(R-2r)\dot{\vartheta}}{2r} \quad (9)$$

We have now obtained that the angular velocity $\dot{\vartheta}$ of rotation of a fixed circular groove on shaft in rotation, in the function of an independent generalized coordinate $\dot{\varphi}$is in the following form:

$$\dot{\vartheta} = 2\frac{(R-r)\dot{\varphi}}{(R-2r)} \quad (10)$$

We see that this angular velocity $\dot{\vartheta}$ is greater than the angular velocity$\dot{\varphi}$ according to the independent generalized coordinate φ.

Fig. 5 shows a pair of balls, each of radius r, and mass density ρ, both at the ends of a diameter of length $2(R-r)$ and geometrical relations of the rolling dynamics, without sliding, of the balls in a radial ball bearing model and in contact with a movable circular groove fitted rigidly to the shaft , which rotates with angular velocity $\dot{\vartheta}$.

Checking the contact connection – kinematical coupling of balls in rolling, non-slip, through pure geometric contact in contact point K and length of arcs crossed in rolling, non-slip, balls in circular paths, one fixed and one movable circular line (see Fig. 5).

Fig. 5 shows the characteristic geometric elements, characteristic points and arcs in correlation, non-slip, heavy homogeneous and isotropic balls in rolling, along a fixed circular path and contact at a point K, without sliding, along the circular contour of the movable circular platform. For the analysis of the geometry and kinematics of rolling and contact kinematics, we adopted two angles, one as an independent generalized coordinate φ and the other dependent coordinate ϑ of rotation of a movable circular platform about its orthogonal axis through its center, in contact, non-slip, with balls, in non-slip rolling path.

The arc P_0P of rolling on a circular fixed path is equal to the arc P_0P on the contour of rolling the ball, while the arc $KP_{0,2}$ of contact on the ball is equal to the arc KK_0 of contact on the circular contour of the moving platform, which rotates about an orthogonal axis on the platform through its center. All this can be seen from the Fig. 5, which also indicates the angle γ for which the rolling ball is rotated, sharpening the characteristics of the arcs. Based on that, we can write:

$$P_0P = PP_{0,1} = KK_0 = KP_{0,2}$$

$$\begin{array}{ll} P_0P = PP_{0,1} & KK_0 = KP_{0,2} \\ R\varphi = r(\varphi+\gamma) & (R-2r)\vartheta - (R-2r)\varphi = r(\varphi+\gamma) \\ \gamma = \frac{(R-r)\varphi}{r} \Rightarrow \omega_P = \dot{\gamma} = \frac{(R-r)\dot{\varphi}}{r} & \gamma = \frac{(R-2r)\vartheta}{2r} \Rightarrow \omega_P = \dot{\gamma} = \frac{(R-2r)\dot{\vartheta}}{2r} \end{array} \quad (11)$$

The following is, also, the ratio of coordinates, which we have obtained before:

$$\gamma = \frac{(R-r)\varphi}{r} = \gamma = \frac{(R-2r)\vartheta}{2r} \quad \Rightarrow \quad \vartheta = 2\frac{(R-r)\varphi}{(R-2r)}$$

$$\vartheta = 2\frac{(R-r)\varphi}{(R-2r)} \quad \text{као и} \quad \varphi = \frac{(R-2r)\vartheta}{2(R-r)} \qquad (12)$$

If the shaft is held at a constant angular velocity, it is estimated that$\varphi = \frac{(R-2r)\Omega t}{2(R-r)}$.

In Fig. 5, the arcs of contact are indicated, which the ball, in rolling, without sliding, describes on the moving and stationary circular groove.

It is indicated that the position of the pair of radial bearing balls is determined by the central angle φ, and also with expressions (7) and (8), and that the angle of rotation of the movable circular groove and the shaft is determined by the angle ϑ.

The last of the expressions (12) gives the kinematic relationship between the angles φ and ϑ, as well as expression (10) between angular velocities, $\dot{\varphi}$ and $\dot{\vartheta}$, of rotation of the hybrid system in the rotor model. The angular velocities of rolling of the balls in the radial ball bearings are defined by expressions (8) or (9), which are mounted in pairs on each of the ends of the rotor shaft, see Fig. 3.

2.3 Ideal case of balanced rotor dynamics and with balanced radial ball bearings

Axial moment of inertia of the mass of a ball and a hollow ball are:

$$\mathbf{J}_0 = \frac{8}{5}\mathbf{M}R^2, \mathbf{J}_x = \mathbf{J}_y = \mathbf{J}_z = \frac{2}{5}\mathbf{M}R^2, \quad \mathbf{M} = \rho\frac{4}{8}R^3\pi, \ \mathbf{J}_x = \mathbf{J}_y = \mathbf{J}_z = \frac{2}{5}\frac{1-\psi^5}{1-\psi^3}\mathbf{M}R^2,$$

$$\mathbf{J}_x = \mathbf{J}_y = \mathbf{J}_z \approx \frac{2}{3}\mathbf{M}R^2, \quad \mathbf{M} \approx 4R^2\pi\delta, \ \mathbf{J}_P = \mathbf{J}_y + \mathbf{M}R^2 == \frac{7}{3}\mathbf{M}R^2 (13)$$

The axial moment $\boldsymbol{J_O}$ of inertia of the mass for the ring and for the shaft axis is: $\boldsymbol{J_O} = \boldsymbol{M}(R-2r)^{2^2}$, $\boldsymbol{M} = \rho 2(R-2r)\pi$, or if it is a grooved disc $\boldsymbol{J_O} = \frac{1}{2}\boldsymbol{M}(R-2r)^2$,
$\mathbf{M} = \rho(R-2r)^2\pi$.

The squares of the instantaneous angular velocity $\omega_P = \frac{(R-r)\dot{\varphi}}{r}$ of the ball rolling along a fixed circular groove and in kinematic contact with a movable circular groove rotating together with the shaft on which the roller bearing is mounted are:

$$(\omega_P)^2 = \left[\frac{(R-r)\dot{\varphi}}{r}\right]^2 \quad \text{and} \quad \dot{\vartheta}^2 = 4\left[\frac{(R-r)\dot{\varphi}}{(R-2r)}\right]^2 = \Omega^2 \ (14)$$

The kinetic energy $\boldsymbol{E_{k,1}}$ of rolling one pair of two balls is: $\boldsymbol{E_{k,1}} = 2\frac{1}{2}\boldsymbol{J_P}(\omega_P)^2 = \frac{1}{2}\boldsymbol{J_P}\left[\frac{(R-r)\dot{\varphi}}{r}\right]^2$.

The kinetic energy $\boldsymbol{E_k}$ of rolling of all eight balls in each of two radial ball bearings and two movable circular grooves with balanced shaft and disks of the axial moment of inertia of the mass $\boldsymbol{J}_O = \frac{1}{2}\boldsymbol{M}(R-2r)^2$, and mass $\boldsymbol{M} = \rho(R-2r)^2\pi$ for the shaft axis and the radial ball bearing is (see Fig. 3.):

$$\mathbf{E_k} = 2\sum_{1}^{8}\frac{1}{2}\mathbf{J_P}(\omega_{P,i})^2 + \frac{1}{2}\mathbf{J}_O(\dot{\vartheta})^2 = \mathbf{J_P}\sum_{i=1}^{i=8}\left[\frac{(R-r)\dot{\varphi}_i}{r}\right]^2 + \frac{1}{2}\mathbf{J}_O 4\left[\frac{(R-r)\dot{\varphi}_1}{(R-2r)}\right]^2 \quad (15)$$

The potential energy $\boldsymbol{E}_p = const$does not change because the center of mass of a pair of balls is in the center of the cross section of the shaft and the roller bearing.

Bearing in mind that the center C of mass of a pair of balls is in the center C of circular, movable and immovable rolling grooves and kinematic contact K of balls in rolling, there is no change in the potential energy $\boldsymbol{E}_p = const$ of the radial ball bearing system in the rolling dynamics, so can write the following ordinary differential equation for all rotate system nonlinear dynamics: By use Lagrange equation of the second kind for generalized coordinate φ: $\frac{d}{dt}\frac{\partial \boldsymbol{E_k}}{\partial \dot{\phi}} - \frac{\partial \boldsymbol{E_k}}{\partial \phi} + \frac{\partial \boldsymbol{E_p}}{\partial \phi_n} = 0$ can be write the following ordinary differential equation:

$$\frac{d}{d}\left\{16\mathbf{J_P}\left[\frac{(R-r)\dot{\varphi}}{r}\right]\frac{(R-r)}{r} + \mathbf{J}_O 4\left[\frac{(R-r)\dot{\varphi}}{(R-2r)}\right]\frac{(R-r)\dot{\varphi}_n}{(R-2r)}\right\} = 0 \quad (16)$$

And the first integral of the previous differential equation is:

$\dot{\varphi} = \dot{\varphi}_0 = const$ and $\dot{\varphi}_n = \dot{\varphi}_{n,0} = const$(17)

Now, it is visible, and can be see that in that case the motion – rollingof balls, without sliding, is a constant instantaneous angular velocity $\omega_{P,n} = \frac{(R-r)\dot{\varphi}_n}{r}$ of rolling, because the change of the independent generalized coordinate - the angle $\dot{\varphi} = \dot{\varphi}_{n,0}t$ is uniform with a constant angular speed$\dot{\varphi} = \dot{\varphi}_{n,0} = const$.

Then, the angular velocity of shaftis $\Omega = \dot{\vartheta} = 2\frac{(R-r)\dot{\varphi}}{(R-2r)} = const$, and an conclude that angular velocity of shaft rotation is constant: $\dot{\vartheta} = 2\frac{(R-r)\dot{\varphi}}{(R-2r)} = \Omega = const$, and also that is$\dot{\varphi} = \dot{\varphi}_{n,0} = \frac{(R-2r)\Omega}{2(R-r)} = const$.

The correct operation of the ball bearing is by inertia.

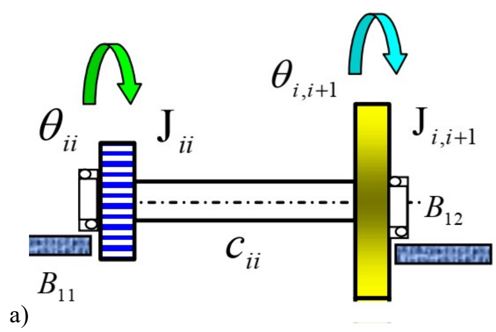
The current angular velocity $\omega_{P,n+1}$ of rolling, withoutslipping, of a pair of balls in a pair on one diameter is:

$$\omega_{P,n+1} = \frac{(R-r)\dot{\varphi}_{n+1}}{r} = \frac{(R-2r)\Omega}{2r} = const \quad (18)$$

3. THE FIRST MODEL: NONLINER DYNAMICS OF THE ROLLING OF THE RADIAL BEARING BALL AND THE NONLINEAR ROTATION OF THE BALANCED ROTOR

Nonlinearities in the dynamics in the ball bearing occur if the balls slide in the rolling, so they are no longer in pairs on the same diameter, but are "tuned" by the position, when there is a change in potential energy and the impact of nonlinearity occurs due to change in potential energy or the appearance of inhomogeneity balls, when the center of mass of the pair of balls"walk in diameter". Rolling balls from pairs is described by a nonlinear differential equation.

The radial bearing balls roll, and balls roll, without slipping, along the stationary circular groove of the radial ball bearing, which is fixed to the foundation. Balls rolling on a stationary circle groove and are in contact, without slipping, with a movable circular groove (on which they "roll" without slipping), which rotates together with the balanced rotor shaft to which it is rigidly connected (see Fig. 3). Angular velocity of the shaft is $\dot{\vartheta}$.



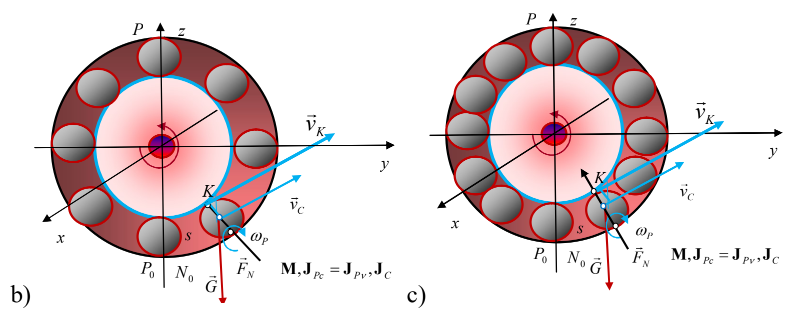


Fig. 6 A rotor with one shaft and two discs, supported by two radial ball bearings in pair (a) with four (b) or six (c) pairs of balls in each pair of bearings

If, during the manufacture of a radial ball bearing, one pair of two balls in the same diameter and of equal spherical surfaces, but of different mass densities of the ball material ρ and $\tilde{\rho} = p\rho$, were mounted, in one pair of balls, on one diameter, then the center of their masses is eccentric in relation to the center of the circular rolling and contact paths and of the dynamic content of rolling balls.

If there is a disturbance in the density $\tilde{\rho} = p\rho$ of the material, a heavy, homogeneous and isotropic ball in a radial ball bearing, and there is no deformation of its spherical boundary surface, as well as no change in its radius r, then the axial mass moment of inertia $\tilde{\boldsymbol{J}}_P$ of that ball for the instantaneous rolling axis is equal to:

$$\tilde{\boldsymbol{J}}_0 = \frac{8}{5}\widetilde{\boldsymbol{M}}R^2, \tilde{\boldsymbol{J}}_x = \tilde{\boldsymbol{J}}_y = \tilde{\boldsymbol{J}}_z = \frac{2}{5}p\boldsymbol{M}R^2, \quad \widetilde{\boldsymbol{M}} = \tilde{\rho}\frac{4}{8}R^3\pi,$$

$$\tilde{\boldsymbol{J}}_P = p\boldsymbol{J}_P = \tilde{\boldsymbol{J}}_y + p\boldsymbol{M}R^2 = \frac{7}{5}p\boldsymbol{M}R^2 \tilde{\boldsymbol{J}}_x = \tilde{\boldsymbol{J}}_y = \tilde{\boldsymbol{J}}_z = \frac{2}{5}\frac{1-\psi^5}{1-\psi^3}p\boldsymbol{M}R^2, \qquad (19)$$

$$\tilde{\boldsymbol{J}}_x = \tilde{\boldsymbol{J}}_y = \tilde{\boldsymbol{J}}_z \approx \frac{2}{3}p\boldsymbol{M}R^2, \widetilde{\boldsymbol{M}} \approx 4p\rho R^2\pi\delta, \tilde{\boldsymbol{J}}_P = \tilde{\boldsymbol{J}}_y + p\boldsymbol{M}R^2 = \frac{7}{3}p\boldsymbol{M}R^2$$

where p is the coefficient of homogeneous change in the material density $\tilde{\rho} = p\rho$ of the ball in the radial ball bearing.

As we observe the balls in pairs and when they are of equal mass density and diameter, the center of mass of one pair of balls is in the center of the concentric circular, stationary and moving, rolling path, that is, dynamic contact.

Although the balls are equal to the spherical boundary surface of the radius r, but of different mass densities, ρ and $\tilde{\rho} = p\rho$, the center of mass of that pair of balls is no in the center of those circular concentric paths, but has an eccentricity: $e = \frac{1}{2}(R - r)\frac{(1-p)}{(1+p)}$, which we determine by determining their center of gravity:

When the mass densities of the balls are equal, the eccentricity is equal to zero.

If the radial ball bearings are in the vertical plane, the eccentricity of the center of mass of the pair of balls when the balls are rolling in the ball circle paths leads to a change in the potential energy and the nonlinear dynamics of the balls rolling.

If the radial ball bearings are in the horizontal plane, the eccentricity of the center of mass of the pair of balls when the balls are rolling in circular paths, leads to the appearance ofchange of centrifugal forces and the nonlinear dynamics of the balls rolling.

In both cases, there is the appearance of kinetic contact forces of pressure on the balls of radial ball bearings, which we will not deal with in this paper.

We will only study the non-linear dynamics of the rotation of a balanced shaft with discs and radial ball bearings, the non-linear phenomenon of non-linear rotation.

If the center of mass of a pair of balls on one diameter has eccentricity $e_n = \frac{1}{2}(R-r)\frac{(1-p_n)}{(1+p_n)}$, $e_n = const, n = 1,2,3,4,(5,6)$, where p_n is the coefficient of variation of the mass density of one of the balls in the pair, then the on-center of mass moves in a circle of radius $e_n = const$ and depending on the angle coordinateφ_n, of balls configuration in bearing, in the vertical direction there is a change in height for $z = e_n(\cos\phi_{n,0} - \cos\phi_n)$, so the change in potential energy is defined by expression:

$$\mathbf{E}_{\mathbf{p},n} = \frac{1}{2}m(1-p_n)g(R-r)(\cos\varphi_{n,0} - \cos\varphi_n) \qquad (20)$$

If all pairs of balls with eccentricity of the center of mass the expression for the potential energy is:

$$\mathbf{E}_{\mathbf{p}} = \sum_{n,1=1}^{n,1=4}\mathbf{E}_{\mathbf{p},n,1} + \sum_{n,2=1}^{n,2=4}\mathbf{E}_{\mathbf{p},n2}$$

$$\mathbf{E_p} = \frac{1}{2}mg(R-r)\left[\sum_{n,1=1}^{n,1=4}(1-p_{n,1})(cos\varphi_{n,1,0}-cos\varphi_{n,1}) + \sum_{n,2=1}^{n,2=4}(1-p_{n,2})(cos\varphi_{n,2,0}-cos\varphi_{n,2})\right]$$

Or in the forms

$$\mathbf{E_p} = \sum_{n,1=1}^{n,1=6}\mathbf{E}_{\mathbf{p},n,1} + \sum_{n,2=1}^{n,2=6}\mathbf{E}_{\mathbf{p},n2}$$

$$\mathbf{E_p} = \frac{1}{2}mg(R-r)\left[\sum_{n,1=1}^{n,1=6}(1-p_{n,1})(cos\varphi_{n,1,0}-cos\varphi_{n,1}) + \sum_{n,2=1}^{n,2=6}(1-p_{n,2})(cos\varphi_{n,2,0}-cos\varphi_{n,2})\right] \quad (21)$$

$$\mathbf{E_p} = \frac{1}{2}mg(R-r)\left[\sum_{n,1=1}^{n,1=N}(1-p_{n,1})(cos\varphi_{n,1,0}-cos\varphi_{n,1}) + \sum_{n,2=1}^{n,2=N}(1-p_{n,2})(cos\varphi_{n,2,0}-cos\varphi_{n,2})\right], N=4,6$$

In the previous expressions, $\varphi_{n,1}$ and $\varphi_{n,2}$ are the contact coordinates of the ball configurations in rolling, without sliding, in radial ball bearings, while $\varphi_{n,1,0}$ and $\varphi_{n,2,0}$ are the contact coordinates of the initial configuration of the balls of the radial ball bearings, on the left (1) and right (2) end of the shaft from Fig. 6.

The squares $(\omega_{P,n})^2$ of the current angular velocities of the balls rolling along a fixed circular groove and in kinematic contact with the movable circular groove that rotates together with the shaft on which the radial roller bearing is mounted are:

$$(\omega_{P,n})^2 = \left[\frac{(R-r)\dot{\varphi}_n}{r}\right]^2 \quad \text{and} \quad (\omega_{P,n+1})^2 = \left[\frac{(R-r)\dot{\varphi}_{n+1}}{r}\right]^2 = \left[\frac{(R-r)\dot{\varphi}_n}{r}\right]^2 \quad (22)$$

The kinetic energy $\boldsymbol{E_{k,n}}$ of rolling a pair of balls, masses of balls of different densities, on one diameter and movable platform with circular grove is:

$$\mathbf{E}_{\mathbf{k},n} = \frac{1}{2}\mathbf{J_P}(1+p_n)(\omega_{P,n})^2 \quad \mathbf{E}_{\mathbf{k},n} = \frac{1}{2}\mathbf{J_P}(1+p_n)\left[\frac{(R-r)\dot{\varphi}_n}{r}\right]^2 \quad (23)$$

The kinetic energy $\boldsymbol{E_k}$of rolling of all eight balls in each of two radial ball bearings, with all unbalanced pair of balls, and two movable platforms with two circular grooves on balanced shaft with two fixed discs on it, is (see Fig. 6):

$$\mathbf{E_k} = \sum_{n,1=1}^{n,1=4}\frac{1}{2}\mathbf{J_P}(1+p_{n,1})(\omega_{P,n,,1})^2 + \sum_{n,2=1}^{n,2=4}\frac{1}{2}\mathbf{J_P}(1+p_{n,2})(\omega_{P,n,2})^2 + \frac{1}{2}\mathbf{J}_O 4\left[\frac{(R-r)\dot{\varphi}_n}{(R-2r)}\right]^2$$

$$\mathbf{E_k} = \frac{1}{2}\mathbf{J_P}\sum_{n,1=1}^{n,1=4}(1+p_{n,1})\left[\frac{(R-r)\dot{\varphi}_{n,1}}{r}\right]^2 + \\ + \frac{1}{2}\mathbf{J_P}\sum_{n,2=1}^{n,2=14}(1+p_{n,2})\left[\frac{(R-r)\dot{\varphi}_{n,2}}{r}\right]^2 + \frac{1}{2}\mathbf{J}_O 4\left[\frac{(R-r)\dot{\varphi}}{(R-2r)}\right]^2 \quad (24)$$

Although for angular velocities $\dot{\varphi}_n = \dot{\varphi}$, it is necessary to take into account the dynamic configuration of the balls.

The dynamic configuration of pairs of two balls can be determined using a single coordinate φ, bearing in mind that the balls are of equal dimensions - the radius of the contour spherical surface, and that they roll, without slipping, in the net, which keeps them at the same distance.

As we focused on structures with 8 (eight) and 12 (twelve) balls, in the first case it is a four pair of balls, and in the second case six pairs of balls. The coordinates of these pairs of balls are previously defined by expressions (6) and (7).

In the case of eight balls, and four unbalanced pairs, two unbalanced balls in each of the pairs, the expression of kinetic and potential energy change of system dynamics of balanced shaft with two discs are (see Fig. 6):

$$\mathbf{E_k} = \frac{1}{2}\mathbf{J_P}\left[\frac{(R-r)\dot{\varphi}}{r}\right]^2 \left\langle \sum_{n,1=1}^{n,1=4}(1+p_{n,1}) + \sum_{n,2=1}^{n,2=4}(1+p_{n,2}) \right\rangle + 2\mathbf{J}_O\left[\frac{(R-r)\dot{\varphi}}{(R-2r)}\right]^2 \tag{25}$$

$$\mathbf{E_p} = \sum_{n=1}^{n=4}\mathbf{E}_{\mathbf{p},n} = \frac{1}{2}mg(R-r)\sum_{n,1=1}^{n,1=4}(1-p_{n,1})(cos\varphi_{n,1,0} - cos\varphi_{n,1}) + \frac{1}{2}mg(R-r)\sum_{n2,=1}^{n,2=4}(1-p_{n,2})(cos\varphi_{n,2,0} - cos\varphi_{n,2})$$

$$\mathbf{E_p} = \sum_{n=1}^{n=4}\mathbf{E}_{\mathbf{p},n} = \frac{1}{2}mg(R-r)\sum_{n,1=1}^{n,1=4}(1-p_{n,1})\left(cos\left(\pi + \frac{(n,1-1)\pi}{4}\right) - cos\left(\varphi + \pi + \frac{(n,1-1)\pi}{4}\right)\right) + \frac{1}{2}mg(R-r)\sum_{n2,=1}^{n,2=4}(1-p_{n,2})\left(cos\left(\pi + \frac{(n,2-1)\pi}{4}\right) - cos\left(\varphi + \pi + \frac{(n,2-1)\pi}{4}\right)\right) \tag{26}$$

In the case of twelve balls, and six unbalanced pairs, two unbalanced balls in each of the pairs, the expression of kinetic and potential energy change of system dynamics of balanced shaft with two discs are (see Figure 6):

$$\mathbf{E_k} = \frac{1}{2}\mathbf{J_P}\left[\frac{(R-r)\dot{\varphi}}{r}\right]^2 \left\langle \sum_{n,1=1}^{n,1=6}(1+p_{n,1}) + \sum_{n,2=1}^{n,2=6}(1+p_{n,2}) \right\rangle + 2\mathbf{J}_O\left[\frac{(R-r)\dot{\varphi}}{(R-2r)}\right]^2 \tag{27}$$

$$\mathbf{E_p} = \sum_{n=1}^{n=4}\mathbf{E}_{\mathbf{p},n} = \frac{1}{2}mg(R-r)\sum_{n,1=1}^{n,1=6}(1-p_{n,1})(cos\varphi_{n,1,0} - cos\varphi_{n,1}) + \frac{1}{2}mg(R-r)\sum_{n2,=1}^{n,2=6}(1-p_{n,2})(cos\varphi_{n,2,0} - cos\varphi_{n,2})$$

$$\mathbf{E_p} = \sum_{n=1}^{n=6}\mathbf{E}_{\mathbf{p},n} = \frac{1}{2}mg(R-r)\sum_{n,1=1}^{n,1=6}(1-p_{n,1})\left(cos\left(\pi + \frac{(n,1-1)\pi}{6}\right) - cos\left(\varphi + \pi + \frac{(n,1-1)\pi}{6}\right)\right) + \frac{1}{2}mg(R-r)\sum_{n2,=1}^{n,2=6}(1-p_{n,2})\left(cos\left(\pi + \frac{(n,2-1)\pi}{6}\right) - cos\left(\varphi + \pi + \frac{(n,2-1)\pi}{6}\right)\right) \tag{28}$$

Or generalized

$$\mathbf{E_k} = \frac{1}{2}\mathbf{J_P}\left[\frac{(R-r)\dot{\varphi}}{r}\right]^2 \left\langle \sum_{n,1=1}^{n,1=N}(1+p_{n,1}) + \sum_{n,2=1}^{n,2=N}(1+p_{n,2}) \right\rangle + 2\mathbf{J}_O\left[\frac{(R-r)\dot{\varphi}}{(R-2r)}\right]^2, \quad N = 4,6 \tag{29}$$

$$\boldsymbol{E_p} = \frac{1}{2}mg(R-r)\sum_{n,1=1}^{n,1=N}(1-p_{n,1})(cos\phi_{n,1,0} - cos\phi_{n,1}) + \frac{1}{2}mg(R-r)N\sum_{n2,=1}^{n,2=4}(1-p_{n,2})(cos\phi_{n,2,0} - cos\phi_{n,2}), \quad N = 4,6$$

$$\mathbf{E_p} = \frac{1}{2}mg(R-r)\sum_{n,1=1}^{n,1=N}(1-p_{n,1})\left(cos\left(\pi + \frac{(n,1-1)\pi}{N}\right) - cos\left(\varphi + \pi + \frac{(n,1-1)\pi}{N}\right)\right) + \frac{1}{2}mg(R-r)\sum_{n2,=1}^{n,2=N}(1-p_{n,2})\left(cos\left(\pi + \frac{(n,2-1)\pi}{N}\right) - cos\left(\varphi + \pi + \frac{(n,2-1)\pi}{N}\right)\right), \quad N = 4,6 \tag{30}$$

The kinetic energy $\boldsymbol{E_k}$ and change of potential energy of the system of two unbalanced radial ball bearings, and with p equal density differences on all pairs of balls, at the ends of a balanced shaft with two disks are:

* In the case of eight balls, and four unbalanced pairs, two balls in each of the pairs, the expression of kinetic and potential energy change are:

$$\mathbf{E_k} = 4\mathbf{J_P}\left[\frac{(R-r)\dot{\varphi}}{r}\right]^2(1+p) + 2\mathbf{J}_O\left[\frac{(R-r)\dot{\varphi}}{(R-2r)}\right]^2 \tag{31}$$

$$\mathbf{E_p} = \sum_{n=1}^{n=4}\mathbf{E}_{\mathbf{p},n} = \frac{1}{2}mg(R-r)(1-p)\sum_{n,1=1}^{n,1=4}(cos\varphi_{n,1,0} - cos\varphi_{n,1}) + \frac{1}{2}mg(R-r)(1-p)\sum_{n2,=1}^{n,2=4}(cos\varphi_{n,2,0} - cos\varphi_{n,2})$$

$$\mathbf{E_p} = \sum_{n=1}^{n=4}\mathbf{E}_{\mathbf{p},n} = \frac{1}{2}mg(R-r)(1-p)\sum_{n,1=1}^{n,1=4}\left(cos\left(\pi + \frac{(n,1-1)\pi}{4}\right) - cos\left(\varphi + \pi + \frac{(n,1-1)\pi}{4}\right)\right) + \frac{1}{2}mg(R-r)(1-p)\sum_{n2,=1}^{n,2=4}\left(cos\left(\pi + \frac{(n,2-1)\pi}{4}\right) - cos\left(\varphi + \pi + \frac{(n,2-1)\pi}{4}\right)\right) \tag{32}$$

* In the case of twelve balls, and six unbalanced pairs, two balls in each of the pairs, the expression of kinetic and potential energy change are:

$$\mathbf{E_k} = 6\mathbf{J_P}\left[\frac{(R-r)\dot{\varphi}}{r}\right]^2(1-p) + 2\mathbf{J}_O\left[\frac{(R-r)\dot{\varphi}}{(R-2r)}\right]^2 \tag{33}$$

$$\mathbf{E_p} = \sum_{n=1}^{n=4} \mathbf{E}_{\mathbf{p},n} = \frac{1}{2} mg(R-r)(1-p) \sum_{n,1=1}^{n,1=6} (cos\varphi_{n,1,0} - cos\varphi_{n,1}) + \frac{1}{2} mg(R-r)(1-p) \sum_{n2,=1}^{n,2=6} (cos\varphi_{n,2,0} - cos\varphi_{n,2})$$

$$\mathbf{E_p} = \sum_{n=1}^{n=6} \mathbf{E}_{\mathbf{p},n} = \frac{1}{2} mg(R-r)(1-p) \sum_{n,1=1}^{n,1=6} \left(cos\left(\pi + \frac{(n,1-1)\pi}{6}\right) - cos\left(\varphi + \pi + \frac{(n,1-1)\pi}{6}\right)\right) + \frac{1}{2} mg(R-r)(1-p) \sum_{n2,=1}^{n,2=6} \left(cos\left(\pi + \frac{(n,2-1)\pi}{6}\right) - cos\left(\varphi + \pi + \frac{(n,2-1)\pi}{6}\right)\right) \quad (34)$$

* in the case of eight (twelve) balls, and four, $N = 4$ (six, $N = 6$) unbalanced pairs, two balls in each of the pairs, the expression of kinetic and potential energy change are:

$$\mathbf{E_k} = N\mathbf{J_P} \left[\frac{(R-r)\dot{\varphi}}{r}\right]^2 (1+p) + 2\mathbf{J}_O \left[\frac{(R-r)\dot{\varphi}}{(R-2r)}\right]^2, \quad n,1; n,2 = N = 4,6 \quad (35)$$

$$\mathbf{E_p} = \frac{1}{2} mg(R-r)(1-p) \sum_{n,1=1}^{n,1=N} (cos\varphi_{n,1,,0} - cos\varphi_{n,1}) + \frac{1}{2} mg(R-r)(1-p) \sum_{n2,=1}^{n,2=N} (cos\varphi_{n,2,0} - cos\varphi_{n,2}), \quad N = 4,6$$

$$\mathbf{E_p} = \frac{1}{2} mg(R-r)(1-p) \sum_{n,1=1}^{n,1=N} \left(cos\left(\pi + \frac{(n,1-1)\pi}{N}\right) - cos\left(\varphi + \pi + \frac{(n,1-1)\pi}{N}\right)\right) + \frac{1}{2} mg(R-r)(1-p) \sum_{n2,=1}^{n,2=N} \left(cos\left(\pi + \frac{(n,2-1)\pi}{N}\right) - cos\left(\varphi + \pi + \frac{(n,2-1)\pi}{N}\right)\right), \quad N = 4,6 \quad (36)$$

If the mass density of the balls is different in only one pair of balls, in both radial ball bearings, at the left and right end of the shaft, but in different positions in the configuration, the expressions of kinetic energy and change in potential energy are:

* In the case of eight balls, and four pairs, two balls in each of the pairs, the expression of kinetic and potential energy change are:

$$\mathbf{E_k} = \mathbf{J_P} \left[\frac{(R-r)\dot{\varphi}}{r}\right]^2 (7+p) + 2\mathbf{J}_O \left[\frac{(R-r)\dot{\varphi}}{(R-2r)}\right]^2 \quad (37)$$

$$\mathbf{E_p} = \frac{1}{2}mg(R-r)(1-p_{n,1})\left(cos\left(\pi+\frac{\pi}{4}\right)-cos\left(\varphi+\pi+\frac{\pi}{4}\right)\right)$$
$$+\frac{1}{2}mg(R-r)(1-p_{n,2})\left(cos\left(\pi+\frac{3\pi}{4}\right)-cos\left(\varphi+\pi+\frac{3\pi}{4}\right)\right) \quad (38)$$

*In the case of twelve balls, and six pairs, two balls in each of the pairs, the expression of kinetic and potential energy change are:

$$\mathbf{E_k} = \mathbf{J_P}\left[\frac{(R-r)\dot{\varphi}}{r}\right]^2(11+p)+2\mathbf{J}_O\left[\frac{(R-r)\dot{\varphi}}{(R-2r)}\right]^2 \quad (39)$$

$$\mathbf{E_p} = \frac{1}{2}mg(R-r)(1-p_{n,1})\left(cos\left(\pi+\frac{\pi}{6}\right)-cos\left(\varphi+\pi+\frac{\pi}{6}\right)\right)$$
$$+\frac{1}{2}mg(R-r)(1-p_{n,2})\left(cos\left(\pi+\frac{5\pi}{6}\right)-cos\left(\varphi+\pi+\frac{5\pi}{6}\right)\right) \quad (40)$$

The nonlinear differential equation of dynamics of a "tuned" pair of two balls on one diameter and in both radial ball bearings, but in configurations $\pi+\frac{\pi}{4}$and $\pi+\frac{3\pi}{4}$, which differ by $\frac{\pi}{2}$, obtained by Lagrange's differential equation of the second kind $\frac{d}{dt}\frac{\partial \boldsymbol{E_k}}{\partial\varphi}-\frac{\partial \boldsymbol{E_k}}{\partial\varphi}+\frac{\partial \boldsymbol{E_p}}{\partial\varphi}=0$, is (see Refs [17-19]):

$$\ddot{\varphi}+\frac{g}{4(R-r)\left[\frac{\mathbf{J_P}}{mr^2}\frac{(7+p)}{(1-p)}+\frac{2\mathbf{J}_O}{m(R-2r)^2(1-p)}\right]}\cdot$$
$$\cdot\left(sin\left(\varphi+\pi+\frac{\pi}{4}\right)+sin\left(\varphi+\pi+\frac{3\pi}{4}\right)\right)=0 \quad (41)$$

This equation is the type of nonlinear differential equation of a mathematical pendulum.

Now let's introduce the characteristic labels in the expressions:

$$\varepsilon_n=\frac{e_n}{(R-r)},\ \mathbf{i}_0^2=\frac{\mathbf{J}_O}{m(R-2r)^2}, \qquad \mathbf{i}_\mathbf{P}^2=\frac{\mathbf{J_P}}{mr^2} \quad (42)$$

$$\lambda=2(R-r)\left[\frac{\mathbf{J_P}}{mr^2}\frac{(7+p)}{(1-p)}+\frac{2\mathbf{J}_O}{m(R-2r)^2(1-p)}\right]=2(R-r)\left[\mathbf{i}_\mathbf{P}^2\frac{(7+p)}{(1-p)}+2\frac{\mathbf{i}_0^2}{(1-p)}\right] \quad (43)$$

$$\frac{g}{\lambda}=\frac{g}{2(R-r)\left[\frac{\mathbf{J_P}}{mr^2}\frac{(7+p)}{(1-p)}+\frac{2\mathbf{J}_O}{m(R-2r)^2(1-p)}\right]}=\frac{g}{2(R-r)\left[\mathbf{i}_\mathbf{P}^2\frac{(7+p)}{(1-p)}+2\frac{\mathbf{i}_0^2}{(1-p)}\right]}$$

and the nonlinear differential equation of dynamics of the systems with unbalanced elements of the radial ball bearings, balls and circular movable platform with balanced shaft and two discs, it can write using the reduced length λ of the radial ball bearing and dimensionless eccentricity ε_n in the form:

$$\ddot{\varphi}_n + \frac{g}{2\lambda}\left(sin\left(\varphi + \frac{3\pi}{4}\right) + sin\left(\varphi + \frac{5\pi}{4}\right)\right) = 0 \quad (44)$$

The nonlinear differential equation of the dynamics of all pairs of balls on all cross-sections in both radial ball bearings, given configurations, and according to expressions (29) and (30) for kinetic energy and change in potential energy, obtained by means of Lagrange's differential equation of the second orders $\frac{d}{dt}\frac{\partial \boldsymbol{E_k}}{\partial \dot{\varphi}} - \frac{\partial \boldsymbol{E_k}}{\partial \varphi} + \frac{\partial \boldsymbol{E_p}}{\partial \varphi} = 0$, is (see Refs [17-19]):

$$\ddot{\varphi} + \frac{g\left\langle \sum_{n,1=1}^{n,1=N}(1-p_{n,1})sin\left(\varphi + \pi + \frac{(n,1-1)\pi}{N}\right) + \sum_{n2,=1}^{n,2=N}(1-p_{n,2})xin\left(\varphi + \pi + \frac{(n,2-1)\pi}{N}\right),\right\rangle}{2(R-r)\left[\frac{\mathbf{J_P}}{mr^2}\left\langle \sum_{n,1=1}^{n,1=N}(1+p_{n,1}) + \sum_{n,2=1}^{n,2=N}(1+p_{n,2})\right\rangle + 4\frac{\mathbf{J}_O}{m(R-2r)}\right]} = 0, \backslash \quad (45)$$

$$N = 4,6$$

where reduced lengths is in the form:

$$\lambda_N = 2(R-r)\left[\frac{\mathbf{J_P}}{mr^2}\left\langle \sum_{n,1=1}^{n,1=N}(1+p_{n,1}) + \sum_{n,2=1}^{n,2=N}(1+p_{n,2})\right\rangle + 4\frac{\mathbf{J}_O}{m(R-2r)}\right], \quad N = 4,6 \quad (46)$$

If we now introduce the following denotation:

* Radius$\boldsymbol{i_P^2}$ of inertia for balls for the current rolling axis$\boldsymbol{i_P^2} = \frac{J_P}{mr^2}$

* The radius $\boldsymbol{i_0^2}$ of the polar moment of inertia for a movable rotating circular platform of radius in relation to the mass of the ball is in the form:$\boldsymbol{i_0^2} = \frac{J_O}{m(R-2r)^2}$, $\boldsymbol{i_P^2} = \frac{J_P}{mr^2}$

* Reduced eccentricity ε_n - dimensionless eccentricity in shape$\varepsilon_n = \frac{e_n}{(R-r)}$.

Then we can also enter the following label and $\frac{g}{\lambda}$ call it the coefficient of the roller bearing, while we will use the following expressions:

$$\lambda_N = 2(R-r)\left[\frac{\mathbf{J_P}}{mr^2}\left\langle \sum_{n,1=1}^{n,1=N}(1+p_{n,1}) + \sum_{n,2=1}^{n,2=N}(1+p_{n,2})\right\rangle + 4\frac{\mathbf{J}_O}{m(R-2r)}\right],$$

$$N = 4,6$$

$$\lambda_N = 2(R-r)\left[\mathbf{i_P^2}\left\langle \sum_{n,1=1}^{n,1=N}(1+p_{n,1}) + \sum_{n,2=1}^{n,2=N}(1+p_{n,2})\right\rangle + 4\mathbf{i_0^2}\right], \quad N = 4,6 \quad (47)$$

and λ name the reduced length and $\kappa = \{2\boldsymbol{i_P^2} + \boldsymbol{i_0^2}\}$ rolling coefficient, of balls and movable platforms in a ball bearing with eight balls, four in a pair of roller bearings.

$\kappa_N = \left[\mathbf{i}_\mathbf{P}^\mathbf{2} \sum_{n=1}^{n=N} (1 + p_n) + 2\mathbf{i}_0^\mathbf{2} \right], \quad N = 4,6$ is rolling coefficient, of balls and movable platforms in a ball bearing with eight balls, four in a pair of roller bearings.

Now the nonlinear differential equation of rolling balls and moving platforms in a ball bearing with eight balls, in four pairs of roller bearings, can be written in the following form:

$$\ddot{\varphi} + \\ + \frac{g}{\lambda_N} \left\langle \sum_{n,1=1}^{n,1=N} (1 - p_{n,1}) sin\left(\varphi + \pi + \frac{(n,1-1)\pi}{N} \right) + \sum_{n2,=1}^{n,2=N} (1 - p_{n,2}) xin\left(\varphi + \pi + \frac{(n,2-1)\pi}{N} \right) \right\rangle = 0, \quad N = 4,6 \tag{48}$$

Singular points of rolling dynamics of balls and movable platforms in a ball bearing with eight balls, in four pairs of roller bearings, can be determined from the conditions in the following form:

$$\frac{d\varphi}{dt} = \dot{\varphi} = 0$$

$$\frac{d\dot{\varphi}}{dt} = -\frac{g}{\lambda_N} \left\langle \sum_{n,1=1}^{n,1=N} (1 - p_{n,1}) sin\left(\varphi + \pi + \frac{(n,1-1)\pi}{N} \right) + \sum_{n2,=1}^{n,2=N} (1 - p_{n,2}) xin\left(\varphi + \pi + \frac{(n,2-1)\pi}{N} \right) \right\rangle = 0, \quad N = 4, = 0 \tag{49}$$

Integral of energy and equations of phase trajectories of a rotor system with two unbalanced two radial ball roller bearings with non-slip roller balls and with two movable circular platforms fixed on a balanced shaft with two disks (see Fig. 6), taking into account expressions (29) for kinetic energy and (30) for change of potential energy, is:

$$\mathbf{E} = \mathbf{E_k} + \mathbf{E_p} = \mathbf{E}_0 - const \tag{50}$$

$$\frac{1}{2}\mathbf{J_P}\left[\frac{(R-r)\dot{\varphi}}{r}\right]^2 \left\langle \sum_{n,1=1}^{n,1=N} (1 + p_{n,1}) + \sum_{n,2=1}^{n,2=N} (1 + p_{n,2}) \right\rangle + 2\mathbf{J}_O \left[\frac{(R-r)\dot{\varphi}}{(R-2r)}\right]^2 + \\ + \frac{1}{2} mg(R-r) \sum_{n,1=1}^{n,1=N} (1 - p_{n,1}) \left(cos\left(\pi + \frac{(n,1-1)\pi}{N} \right) - cos\left(\varphi + \pi + \frac{(n,1-1)\pi}{N} \right) \right) \\ + \frac{1}{2} mg(R-r) \sum_{n2,=1}^{n,2=N} (1 - p_{n,2}) \left(cos\left(\pi + \frac{(n,2-1)\pi}{N} \right) - cos\left(\varphi = +\pi + \frac{(n,2-1)\pi}{N} \right) \right) \\ = \mathbf{E}_0 = const, \quad N = 4,6$$

The equations of phase trajectories of nonlinear dynamics of the considered rotor system with balanced shaft with two discs and two unbalanced radial ball bearings is:

$$\dot{\varphi}^2 = \dot{\varphi}_0^2 - \\ -\frac{g}{\lambda_N}\sum_{n,1=1}^{n,1=N}(1-p_{n,1})\left(cos\left(\varphi_{1,0}+\pi+\frac{(n,1-1)\pi}{N}\right)-cos\left(\varphi+\pi+\frac{(n,1-1)\pi}{N}\right)\right) \\ -\frac{g}{\lambda_N}\sum_{n2,=1}^{n,2=N}(1-p_{n,2})\left(cos\left(\varphi_{2,0}+\pi+\frac{(n,2-1)\pi}{N}\right)-cos\left(\varphi=+\pi+\frac{(n,2-1)\pi}{N}\right)\right) \\ N=4,6 \tag{51}$$

By changing the initial conditions, the initial angular velocity $\dot{\varphi}_0 = \dot{\varphi}(t=0)$ and the initial angular coordinates $\varphi_0 = \varphi(t=0)$, a series of phase trajectories can be drawn and with the help of them a phase portrait of the nonlinear dynamics of the rotor system with balanced radial ball bearings can be constructed.

4. THE SECOND MODEL: NONLINER DYNAMICS OF ROLLING BALLS IN THE RADIAL BALL BEARING AND THE NONLINEAR ROTATION OF THE UNBALANCED ROTOR

In this part, we will study the nonlinear dynamics of rotation of a multi-stage rotary transmission model by reducing the model with multiple shafts and on each two unbalanced discs to the first shaft according to the reduction method presented by D. Rašković in the monograph [17], (see References [17-19] and Fig.s 7 and 8).

The transmission ratio, in fact, is defined by the ratio of the radius of the discs (gears) in engagement, from the next and from the previous shaft, in the form:

$$k_{i(i+1)} = \frac{R_{(i+1)1}}{R_{i2}} = \frac{1}{i_{i(i+1)}} \tag{52}$$

Apropos

$$\dot{\vartheta}_k = i_{1k}\dot{\vartheta}_1 = \prod_{j=1}^{j=k} i_{j(j+1)}\dot{\vartheta}_1 = \frac{1}{k_{1k}}\dot{\vartheta}_1 = \prod_{j=1}^{j=k}\frac{1}{k_{j(j+1)}}\dot{\vartheta}_1 = \prod_{j=1}^{j=k}\frac{1}{k_{j(j+1)}}\omega_1 \tag{53}$$

It is assume that each of the discs can be represented by two elements, a basic part which is balanced in the form of a circular disc of contour with the center of mass exactly on the shaft and in a section in which it is rigidly fixed to the shaft and with a basic axial moment of inertia about the axis of the shaft, $\boldsymbol{J}_{O11}$ and $\boldsymbol{J}_{O12}$, and by unbalance in the form of the material point of mass $m_{kp}, k = 1,2,\dots,N; p = 1,2$and at the distance $r_{kp}, k = 1,2,\dots,N; p = 1,2$from the axis of the shaft for the first and second disc on each shaft.

The axial moment of inertia of each of the shafts with the discs and the unbalanced material points on each of the two discs is:

$$\mathbf{J}_1 = 2\mathbf{J}_{\mathbf{P}}\sum_{i=1}^{i=8}\left[\frac{(R-2r)}{r}\right]^2 + \left[4\widetilde{\mathbf{J}}_{O1} + \frac{1}{2}\left(m_{11}r_{11}^2 + m_{12}r_{12}^2\right)\right], \widetilde{\mathbf{J}}_{O1} = \mathbf{J}_O + \mathbf{J}_{O11} + \mathbf{J}_{O12} \tag{54}$$

$$\mathbf{J}_2 = 2\mathbf{J}_\mathbf{P}\sum_{i=1}^{i=8}\left[\frac{(R-2r)}{r}\right]^2 + \left[4\widetilde{\mathbf{J}}_{O2} + \frac{1}{2}\left(m_{21}r_{21}^2 + m_{22}r_{22}^2\right)\right],\ \widetilde{\mathbf{J}}_{O2} = \mathbf{J}_O + \mathbf{J}_{O21} + \mathbf{J}_{O22} \quad (55)$$

$$\mathbf{J}_3 = 2\mathbf{J}_\mathbf{P}\sum_{i=1}^{i=8}\left[\frac{(R-2r)}{r}\right]^2 + \left[4\widetilde{\mathbf{J}}_{O3} + \frac{1}{2}\left(m_{31}r_{31}^2 + m_{32}r_{32}^2\right)\right],\ \widetilde{\mathbf{J}}_{O3} = \mathbf{J}_O + \mathbf{J}_{O31} + \mathbf{J}_{O3} \quad (56)$$

and so on, for subsequent shafts and a multi-stage rotation transmission, with several coupled shafts connected by disks, that is, gears.

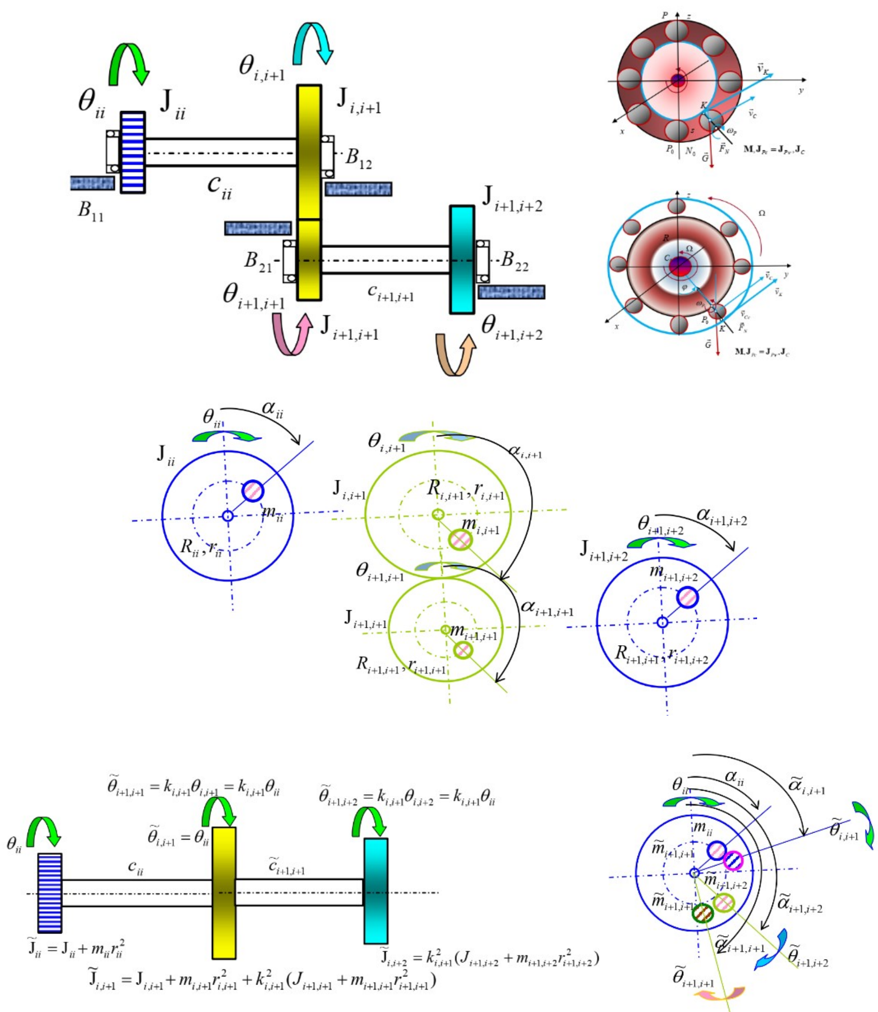


Fig.7 Reduction of the model of the two-stage gear transfer of rotationwith two shafts, to

a model with one shaft

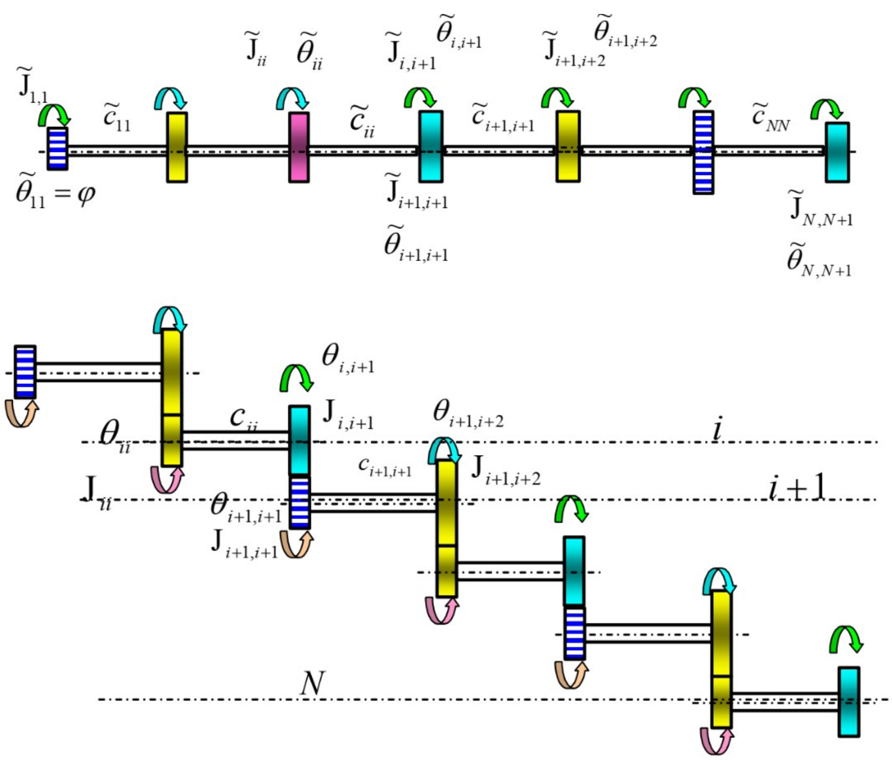


Fig. 8 Reduction of the multi-sstep transnission of the rotation with mani shafts to model with one shaft, with milti gear model to a single-shaft model

Now, I will present the method of reducing the nonlinear dynamics of the rotation transmission system from unbalanced series of the shafts to the first rotating shaft (see Figs 7 and 8), which I chose as the basic one. And the angular coordinate ϑ_1 of rotation of the first shaft, in a series of multi-stage rotation transmission, was chosen as a generalized coordinate, using previously defined axial moments of inertia of balanced and unbalanced elements - inertia parameters (54)-(56) of the shaft respectively in conjunction. This is shown visually graphically, in Fig. 7 for a two-stage and Fig. 8 for a multi-stage gearbox, and for the corresponding transmission ratios (52) and (53).

The expressions for the kinetic energy and the change in potential energy of the two-stage rotation transmission, with two coupled unbalanced shafts, reduced to the first shaft and expressed using the chosen generalized coordinate ϑ_1, the real angle of rotation of the first shaft, are:

$$\boldsymbol{E}_{\boldsymbol{k},two} = \frac{1}{2}\boldsymbol{J}_1\left(\dot{\vartheta}_1\right)^2 + \frac{1}{2}\boldsymbol{J}_2\left(i_{1,2}\dot{\vartheta}_1\right)^2 = \frac{1}{2}(\boldsymbol{J}_1 + i_{12}^2\boldsymbol{J}_2)\dot{\vartheta}_1^2 \quad (57)$$

$$\mathbf{E}_{P,two} = g\left[\left(m_{11}r_{11}+m_{12}r_{12}\right)\left(1-\cos\vartheta_1\right)+\left(m_{21}r_{11}+m_{22}r_{22}\right)\left(1-\cos\left(i_{1,2}\vartheta_1\right)\right)\right] \quad (58)$$

The non-linear differential equation of rotation of a two-step transmission of rotation, with two coupled unbalanced shafts, each with two unbalanced discs, is of the form:

$$\ddot{\vartheta} + \frac{g}{\left(\mathbf{J}_1 + i_{12}^2\mathbf{J}_2\right)}\left[\left(m_{11}r_{11}+m_{12}r_{12}\right)\sin\vartheta_1 + i_{1,2}\left(m_{21}r_{11}+m_{22}r_{22}\right)\sin\left(i_{1,2}\vartheta_1\right)\right] = 0 \quad (59)$$

The singular points of this nonlinear dynamics of rotation, with two coupled shafts, each with two unbalanced disks, are obtained from the following results:

$$\frac{d\boldsymbol{\vartheta}}{dt} = \dot{\boldsymbol{\vartheta}} = \mathbf{0}$$

$$\frac{d\dot{\vartheta}}{dt} = -\frac{g}{\left(\mathbf{J}_1 + i_{12}^2\mathbf{J}_2\right)}\cdot \quad (60)$$
$$\cdot\left[\left(m_{11}r_{11}+m_{12}r_{12}\right)\sin\vartheta_1 + i_{1,2}\left(m_{21}r_{11}+m_{22}r_{22}\right)\sin\left(i_{1,2}\vartheta_1\right)\right] = 0,$$

The nonlinear equation of the phase trajectories, the nonlinear dynamics of a two-stage rotation transmission, with two coupled shafts, each with two unbalanced disks, is obtained from the energy integral and itcan write it in the form:

$$\dot{\vartheta}_1^2 = \dot{\vartheta}_{1,0}^2 + \frac{2g}{\left(\mathbf{J}_1 + i_{12}^2\mathbf{J}_2\right)}\cdot \quad (61)$$
$$\cdot\left[\left(m_{11}r_{11}+m_{12}r_{12}\right)\left(\cos\vartheta_1 - \cos\vartheta_{1,0}\right)+\left(m_{21}r_{11}+m_{22}r_{22}\right)\left(\cos\left(i_{1,2}\vartheta_1\right)-\cos\left(i_{1,2}\vartheta_{1,0}\right)\right)\right]$$

The expressions for the kinetic energy and the change in potential energy of the three-stage rotation transmission reduced to the first shaft and expressed using the chosen generalized coordinate ϑ_1, the real angle of rotation of the first shaft, are:

$$\mathbf{E}_{\mathbf{k},three} = \frac{1}{2}\mathbf{J}_1\left(\dot{\vartheta}_1\right)^2 + \frac{1}{2}\mathbf{J}_2\left(i_{1,2}\dot{\vartheta}_1\right)^2 + \frac{1}{2}\mathbf{J}_3\left(i_{2,3}i_{2,3}\dot{\vartheta}_1\right)^2 = \frac{1}{2}\left(\mathbf{J}_1 + i_{12}^2\mathbf{J}_2 + \left(i_{2,3}i_{2,3}\right)^2\mathbf{J}_3\right)\dot{\vartheta}_1^2 \quad (62)$$

$$\mathbf{E}_{P,threme} =$$
$$= g\left[\left(m_{11}r_{11}+m_{12}r_{12}\right)\left(1-\cos\vartheta_1\right)+\left(m_{21}r_{11}+m_{22}r_{22}\right)\left(1-\cos\left(i_{1,2}\vartheta_1\right)\right)\right]+ \quad (63)$$
$$+ g\left[\left(m_{31}r_{31}+m_{32}r_{32}\right)\left(1-\cos\left(i_{2,3}i_{2,3}\vartheta_1\right)\right)\right]$$

The non-linear differential equation of the non-linear rotary shaft of the three-steps transmission of the rotation, with three coupled shafts each with two unbalanced disks, is:

$$\ddot{\vartheta} + \quad (64)$$
$$+\frac{g}{\left(\mathbf{J}_1 + i_{12}^2\mathbf{J}_2 + \left(i_{2,3}i_{2,3}\right)^2\mathbf{J}_3\right)}\left[\left(m_{11}r_{11}+m_{12}r_{12}\right)\sin\vartheta_1 + i_{2,3}\left(m_{21}r_{11}+m_{22}r_{22}\right)\sin\left(i_{1,2}\vartheta_1\right)\right]+$$
$$+\frac{g}{\left(\mathbf{J}_1 + i_{12}^2\mathbf{J}_2 + \left(i_{2,3}i_{2,3}\right)^2\mathbf{J}_3\right)}\left[i_{2,3}i_{2,3}\left(m_{31}r_{31}+m_{32}r_{32}\right)\sin\left(i_{2,3}i_{2,3}\vartheta_1\right)\right] = 0$$

The singular points of this nonlinear dynamics of first shaft rotation of the three-steps transmission of the rotation, with three coupled shafts each with two unbalanced disks, are obtained from the following results:

$$\frac{d\vartheta}{dt}=\dot{\vartheta}=0$$

$$\frac{d\dot{\vartheta}}{dt}=-\frac{g}{\left(\mathbf{J}_1+i_{12}^2\mathbf{J}_2+\left(i_{2,3}i_{2,3}\right)^2\mathbf{J}_3\right)}\left[\left(m_{11}r_{11}+m_{12}r_{12}\right)sin\,\vartheta_1+i_{2,3}\left(m_{21}r_{11}+m_{22}r_{22}\right)sin\left(i_{1,2}\vartheta_1\right)\right]-$$
$$-\frac{g}{\left(\mathbf{J}_1+i_{12}^2\mathbf{J}_2+\left(i_{2,3}i_{2,3}\right)^2\mathbf{J}_3\right)}\left[i_{2,3}i_{2,3}\left(m_{31}r_{31}+m_{32}r_{32}\right)sin\left(i_{2,3}i_{2,3}\vartheta_1\right)\right]=0, \quad (65)$$

The nonlinear equation of the phase trajectories of nonlinear rotation of a three-stage rotation transmission, with three coupled shafts each with two unweighted disks, is obtained from the energy integral, and we can write it in the form:

$$\dot{\vartheta}_1^2=\dot{\vartheta}_{1,0}^2++\frac{2g}{\left(\mathbf{J}_1+i_{12}^2\mathbf{J}_2+\left(i_{2,3}i_{2,3}\right)^2\mathbf{J}_3\right)}\left[\left(m_{11}r_{11}+m_{12}r_{12}\right)\left(cos\,\vartheta_1-cos\,\vartheta_{1,0}\right)\right]+$$
$$+\frac{2g}{\left(\mathbf{J}_1+i_{12}^2\mathbf{J}_2+\left(i_{2,3}i_{2,3}\right)^2\mathbf{J}_3\right)}\left[\left(m_{21}r_{11}+m_{22}r_{22}\right)\left(cos\left(i_{1,2}\vartheta_1\right)-cos\left(i_{1,2}\vartheta_{1,0}\right)\right)\right]+ \quad (66)$$
$$+\frac{2g}{\left(\mathbf{J}_1+i_{12}^2\mathbf{J}_2+\left(i_{2,3}i_{2,3}\right)^2\mathbf{J}_3\right)}\left[\left(m_{31}r_{31}+m_{32}r_{32}\right)\left(cos\left(i_{2,3}i_{2,3}\vartheta_1\right)-cos\left(i_{2,3}i_{2,3}\vartheta_{1,0}\right)\right)\right]$$

In following we write the corresponding analytical descriptions for a multi-steps transmission of rotation, with several coupled shafts, each of which carries two unbalanced discs, reduced to the first shaft and using the generalized angular coordinateϑ_1 of the rotation of the first shaft.

That's why we introduce simplifications, assuming homogeneity of the system.

For the case of equal unbalances on each of the shafts $\left(m_{11}r_{11}+m_{12}r_{12}\right)=\left(m_{k1}r_{k1}+m_{k2}r_{k2}\right)=2mr_e=const$ and that the transmission ratio from one to the next shaft is equal to the previous one, , $k_{i,i+1}=\frac{R_{i+1}}{R_{i2}}=\frac{1}{i_{i,i+1}}=k=\frac{1}{i}$, as well as that the basic balanced structures on each of the shafts in the structure system are equal, we can simplify the equation of the phase trajectory and write it for a multi-stage rotation transmission. With this simplification, we do not lose the variety of nonlinear phenomena of the nonlinear dynamics of rotation, and the variety of singular points and the specificity and variety of forms of phase trajectories, and the existence of triggers of coupled s singularities.

Now let's introduce the following denotations:

$$\mathbf{J}_{M,M}=\sum_{k=1}^{k=M}i^{2(k-1)}\mathbf{J}_k\,, \qquad \frac{4g}{\lambda}=\frac{2g2mr_e}{\sum_{k=1}^{k=M}i^{2(k-1)}\mathbf{J}_k}=\frac{4gmr_e}{\mathbf{J}_{M,M}}$$

$$\lambda = \frac{\mathbf{J}_{M,M}}{mr_e} = r_e \mathbf{i_{MM}}^2, \qquad \mathbf{i_{MM}}^2 = \frac{\mathbf{J}_{M,M}}{nr_e^2} = \frac{\sum_{k=1}^{k=M} i^{2(k-1)}\mathbf{J}_k}{nr_e^2} \quad (67)$$

The nonlinear equation of the phase trajectory of the nonlinear dynamics of the rotation M- step transmission of rotary motion, with Mcoupled shafts with the same transmission ratio and homogeneous structure, is of the form:

$$\dot{\vartheta}_1^2 = \dot{\vartheta}_{1,0}^2 + \frac{4g}{\lambda}\sum_{k=1}^{k=M}\left(cos\left(i^{k-1}\vartheta_1\right) - cos\left(i^{k-1}\vartheta_{1,0}\right)\right), \quad (68)$$

The angular rolling velocity of balls in rolling without slipping, in radial ball bearings, on the shaft with nonlinear dynamics caused by unbalanced disks on shafts, are of the form:

$$\omega_P = \frac{(R-2r)\dot{\vartheta}}{2r}$$

$$\omega_P = \frac{(R-2r)\dot{\vartheta}}{2r} = \frac{(R-2r)}{2r}\sqrt{\dot{\vartheta}_{1,0}^2 + \frac{4g}{\lambda}\sum_{k=1}^{k=M}\left(cos\left(i^{k-1}\vartheta_1\right) - cos\left(i^{k-1}\vartheta_{1,0}\right)\right)} \quad (69)$$

As the rotation of the shaft is non-linear, caused by the unbalanced discs on them, so are the rolling angular velocities of the non-linear function of time and angular coordinates.

This previous equation (69) is also the equation of the phase trajectory of the non-linear dynamics of rolling, without sliding, of a ball in a radial ball bearing, although all radial ball bearings in a multi-stage rotary transmission are ideally made and mounted. Shaft imbalance causes non-linear ball dynamics in radial ball bearings.

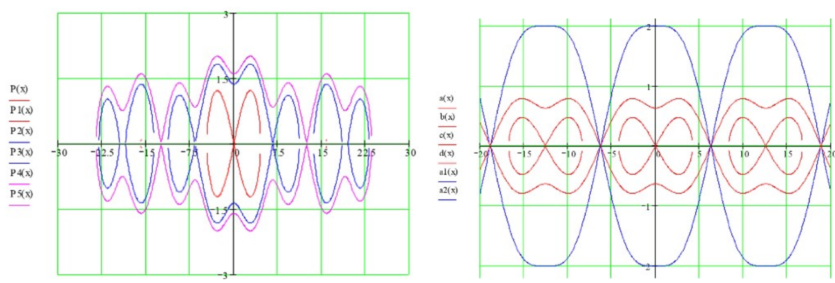
The phase portrait of the rolling balls is analogous to the phase portrait of the nonlinear rotation of the unbalanced coupled shafts of the multiple level transmission system.

A scheme of disks with imbalances in the form of material points is also shown, and the angular velocity of rotation of each disk is indicated. Those angular velocities are different, which depends on the transmission ratio of the disks in contact, as well as on the position of the imbalance (material points) on a particular disk. This leads to the non-linear dynamics of the rotor rotation, and thus to the non-linear dynamics of the rolling of the balls in the radial ball bearings.

5. PHASE TRAJECTORIES AND PHASE PORTRAITS OF NONLINEAR ROTOR DYNAMICS

In this part of the paper, we will present several characteristic phase portraitsof the nonlinear dynamics of a multi-stage rotation transmission, homogeneousstructure. We will show several characteristic phase homoclinic trajectories in particular and give an analysis of the singularity structure of the phase portraits of the nonlinear dynamics of rotation of multi-stage rotation transmissions with shafts, which carry unbalanced discs.

Using mathematical, qualitative and structural analogies, and mathematical phenomenology, we show that the phase portraits of the nonlinear dynamics of rolling balls of radial ball bearings are analogous to the phase portraits of the nonlinear dynamics of rotation of a multi-stage power transmission of rotation. There are also a mathematical and structural analogies of the phase portraits of the nonlinear dynamics of the model of the multi-stage rotation transmission, when the shafts with discs are balanced, and the pairs of balls, on the same diameters of the radial ball bearings are unbalanced, and vice versa.

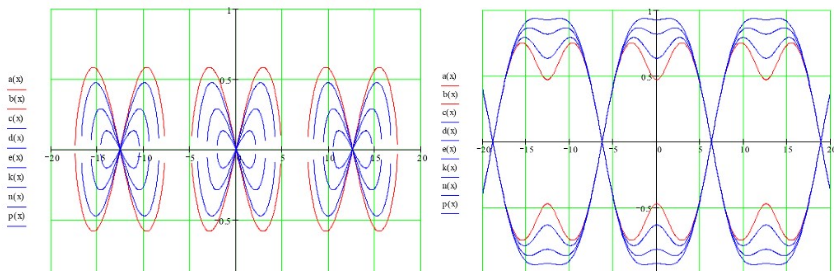


a) b)

Fig. 9 Characteristic phase trajectories - homoclinic orbits with self-intersection in singular points of the unstable saddle type, which contain and include two singular points of the type of stable centers with the trigger phenomenon of coupled singularities extracted from the phase portrait of the nonlinear dynamics of coupled rotations of coupled shafts of multi-stage transmission rotation

In Fig. 9, characteristic phase trajectories - homoclinic orbits are presented, which are with self-intersect in singular points of the unstable saddle type.

They contain a trigger of the coupled three singular points (or a larger odd number of singular points: five, seven, nine, eleven, etc.) and include two singular points of the type of stable centers and form with them the phenomenon of trigger of coupled singularities (or a larger odd number of singularities points: five, seven, nine, eleven, etc., alternately stable singular points of the type of centers or unstable singular points of the type of unstable seats in which the phase trajectories of the lower level only intersect), extracted from the phase portrait of the nonlinear dynamics of coupled rotations of the multi-level transmission of rotation.

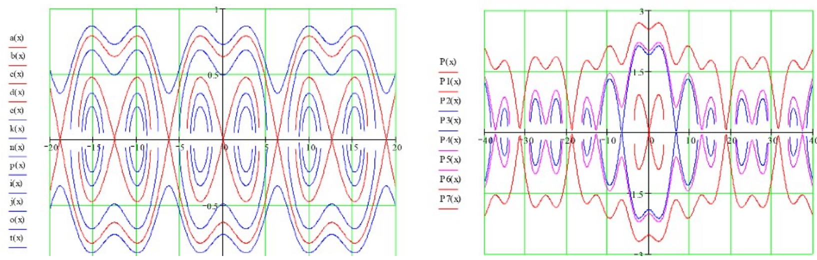


a) b)

Fig. 10 Layering of characteristic phase trajectories - homoclinic trajectories from the phase portrait of the nonlinear dynamics of coupled shaft rotations of a multi-stage transmission of rotation: a) bifurcation of the singular point of the stable center type into three singular points, passing into the singular point of the unstable saddle type in which the phase trajectory self-intersects, and the appearance, around it, of two singular points of the type of stable centers and the layering of that homoclinic phase trajectory in the form of the number "eight", which includes the trigger formed by the bifurcation of those singularities; b) delamination of one separatrix by changing the parameters of the two-stage transfer of rotations

Fig. 10shows the layering of the characteristic phase trajectories - homoclinic trajectories with the change of the system parameters, and especially the influence of the change of the bifurcation parameters, extracted from the phase portrait of the nonlinear dynamics of the coupled shaft rotations of the multi-stage transmission of rotation. Figure 10a shows the phenomenon of bifurcation of a singular point of the stable center type, by changing the bifurcation parameter, and the formation of three singular points, passing into a singular point of the unstable saddle type and with the appearance, around it, of two new singular points of the type of stable centers and the appearance of a homoclinic phase trajectory in the form of the number "eight" and the layering of that homoclinic phase trajectory in the form of the number "eight".

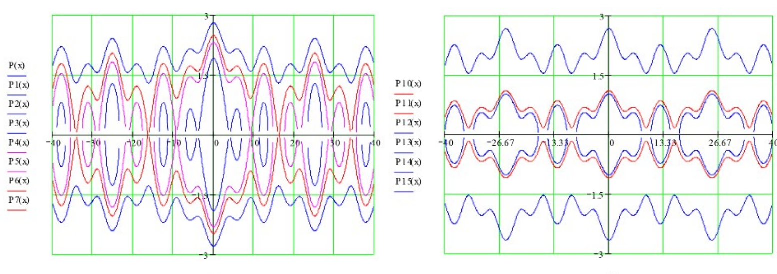
It includes the trigger formed by the bifurcation of those three singular points, forming with them a trigger of coupled singularities. Fig. 10b) shows the delamination of a separatrix, of a higher level, also with self-intersection, which contains a trigger of coupled singularities, and a change in the parameters of the two-stage transfer of rotation. It includes a trigger of coupled singularities, together with a homoclinic phase trajectory in the form of the number "eight".



a) b)

Fig.11 Phase portraits of the nonlinear dynamics model of two multi-stage gearbox with rotating multiple coupled shafts with unbalanced discs and balanced radial ball bearings: a) two-stage gearbox with balanced radial ball bearings on both shafts with unbalanced discs, which carry two unbalanced disc; b) three-stage transmission of rotation with balsamized radial ball bearings and with two, on each of the three coupled shafts, carrying two unbalanced discs

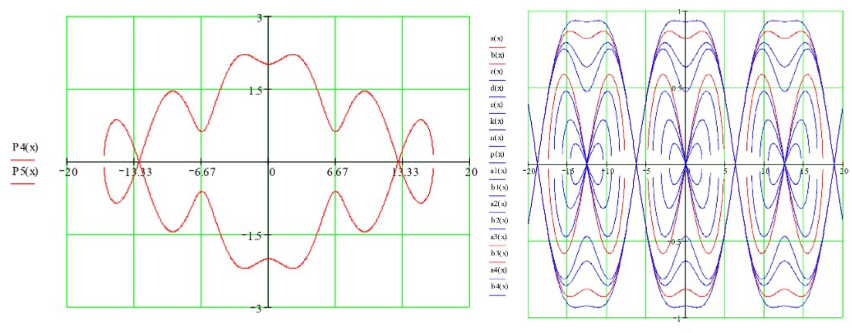
Figs 11and 12 shows the characteristic phase portraits of the nonlinear dynamics of the model of a multi-stage transmission of rotation of a number of coupled shafts with unbalanced shafts and balanced radial ball bearings. Fig. 11a) shows the phase portrait of the nonlinear dynamics of a two-stage rotary transmission system with balanced radial ball bearings, two on each shaft, carrying two unbalanced discs (gears). Fig. 11 b) shows a phase portrait of the nonlinear dynamics of a three-speed rotary transmission, with balsamized radial ball bearings, two on each of three coupled shafts, carrying two unbalanced discs.



a) b)

Fig. 12 Phase portraits of the nonlinear dynamics model of two multi-stage gearbox-transmission of rotation rotating multiple coupled shafts with unbalanced discs and balanced radial ball bearings: a) phase portrait of a four-stage gearbox with balanced radial ball bearings and two unbalanced discs on each of the four shafts, which carry two unbalanced disks each; b) characteristic phase trajectories of a three-stage rotation transmission, with balanced radial ball bearings, two on each of the three shafts, which carry two unbalanced discs on each of the shafts

Fig. 13a)show one of the characteristic homoclinic phase trajectories, from the phase portrait of the nonlinear dynamics of the model of the four-stage transmission of rotation, in which four shafts are coupled. That homoclinic phase trajectory self-intersects in two singular points of the unstable saddle type, and includes eleven singular points, five types of unstable saddle types and six types of stable centers. It contains and surrounds a central trigger of coupled singularities with a phase trajectory in the shape of the number "eight". It also surrounds another separation line-homoclinic phase trajectory, that self-intersects at two singular points and includes a total of five singular points, including the described trigger of coupled singularities with a phase trajectory of the shape of the number "eight". It means that, attached in Fig. 13a), is the homoclinic phase trajectory of the third level, and it contains and surrounds those homoclinic phase trajectories of the lower level: the first level with one trigger of coupled singularities with three singular points, and the second level, which include seven singular points.



a) b)

Fig. 13 Bifurcation and delamination of characteristic phase trajectories - homoclinic trajectories of the nonlinear dynamics of a multi-stage rotation transmission model, with multiple coupled shafts with unbalanced discs and balanced radial ball bearings: a) Homoclinic phase trajectory of a four-stage rotation transmission, which is self-intersecting in two singular points, of the type of unstable saddles, and eleven singular points are surrounded by them, five singular points of unstable saddle types and six singular points types of stable centers, and it contains and surrounds one central trigger of coupled singularities with a phase trajectory in the form of the number "eight"; b) delamination of two separatrices from phase portrait of the nonlinear dynamics of a two-stage rotation transmission with two coupled shafts with balanced radial ball bearings on both two ends of each of two shafts, each of which carries two unbalanced discs and the bifurcation of a singular point of the stable center type into three coupled singular points, passing into a singular point of the unstable saddle type and the appearance, around it, of two singular points of the type of stable centers and the delamination of the homoclinic orbit in the form of the number "eight", which includes the trigger formed by the bifurcation of those coupled singular points

Fig. 13 b)show the delamination of two characteristic separatrices (homoclinic phase trajectories) of the nonlinear dynamics of a two-stage rotation transmission, with two coupled shafts, each of which carries two unbalanced discs, and with balanced radial ball bearings on the ach of bout of two ends of the two shafts. The picture also show the bifurcation of a singular point, of the stable center type, into three singular points, passing into a singular point of the unstable saddle type. The loss of that stability was accompanied by the appearance around it of two new singular points of the type of stable centers. The bifurcation is followed by the delamination of the homoclinic orbit in the form of the number "eight", which includes the trigger formed by the bifurcation of those coupled singular points.

Here we have presented only a small part of the characteristic phase portraits, self-intersecting homoclinic phase trajectories, which contain information about the structure of singular points and the triggers of coupled singularities of the nonlinear dynamics of multi-stage, with multi-shafts coupled, and carried by unbalanced disks and supported on balanced radial ball bearings.

The goal was to bring from these graphical representations a qualitative picture and properties of the qualitative properties of the nonlinear dynamics of the studied models of multi-stage rotary transmissions, coupled with multiple shafts, carrying unbalanced discs and supported on balanced radial ball bearings.

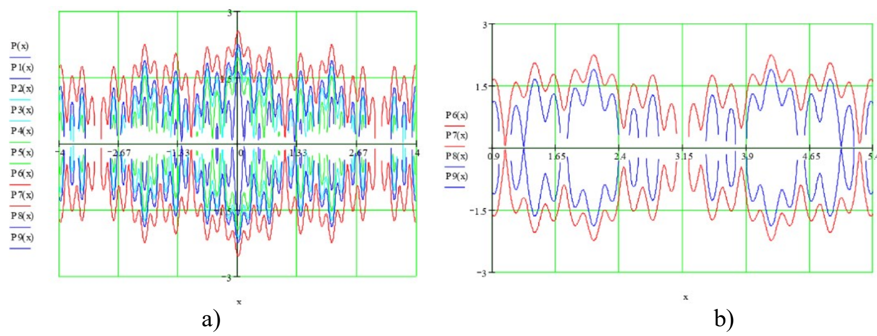


Fig. 14 Phase portraits of the nonlinear dynamics of the model of a multi-stage multi-coupling shaft gearbox with unbalanced discs and balanced radial ball bearings at a higher number of revolutions and a longer time interval

Figs 14a) and 14b) show the phase portraits of the nonlinear dynamics of a model of two multi-stage gear transmission of rotations, with multiple coupled shafts and unbalanced discs and supported on balanced radial ball bearings at a greater number of revolutions and in a longer time interval.

By identifying the elements of mathematical, qualitative and structural analogy and mathematical phenomenology (see References [13-16]), based on mathematical descriptions and derived corresponding ordinary nonlinear differential equations of rotation and equations of phase trajectories of non-linear dynamics of rotation of multi-stage rotation transmissions, 1* when coupled shafts with disks are balanced, and pairs of rolling balls are unbalanced in radial ball bearings and 2* when the coupled shafts with disks are unbalanced, and all pairs of rolling balls of radial ball bearings are balanced, we can conclude that exists also and analogy of phase portraits, as well as all other properties of phase trajectories, structures of singular points and that the qualitative properties of nonlinear the dynamics of one model can be transferred to another model by analogy. The number of pairs of balls with center-of-mass eccentricities corresponds to the number of coupled shafts with unbalanced disks, in consideration of the analogy of phase portraits.

6. CONCLUDING REMARKS

The originality of these models of nonlinear dynamics of rotors and balls of radial ball bearings lies in their analytical contribution and in them enabling qualitative analysis of nonlinear dynamics, which is significant for engineering practice, as well as for teaching machine theory and theoretical dynamics in general. The abstract and theintroduction present a short review of the author's recent research into the topic of the theoretical models of nonlinear dynamic hybrid systems containing the dynamics of radial ball bearings and unbalanced multi-step redactor/multiplier power transmission. The analysis and results should be a theoretical reference for engineering designs of hybrid systems with a bearing and multi-step power transmission nonlinear dynamics.

For the case when the rotor is unbalanced, and the centers of masses of both radial ball bearings have no eccentricity, the nonlinearity of the rotation of the shafts, as well as the rolling of the balls of the radial ball bearing, originates from that unbalance of the multi-step rotor.

Singular points were determined for the nonlinear dynamics of both mentioned models, and for special cases of a multi-stage unbalanced rotor, a series of phase portraits was drawn and the appearance of bifurcation and triggers of coupled singularities and coupled triggers of coupled singularities were analyzed.

It was shown that mathematical and structural analogy and mathematical phenomenology are very useful methods and that by studying one model of nonlinear dynamics, the knowledge obtained can be easily transferred to a completely different physically disparate model, which describes disparate sources and causes of nonlinearity and nonlinear dynamics.

We also showed that the graphical representations of nonlinear phenomena through phase portraits of the nonlinear dynamics of rolling balls, without sliding, in radial ball bearings on balanced shafts are analogous to the phase portraits of the nonlinear dynamics of rotation of a reduced multi-stage model of transmission of rotation and power transmission, on a single shaft founded on balanced radial ball bearings. Also, the resultsshowed that there is a mathematical and structural analogy of the phase portraits of the nonlinear dynamics of a multi-stage transmission system of rotation, when the shafts with disks are unbalanced, and the pairs of balls, in rolling, of radial ball bearings are balanced, and vice versa.

The structural analogy of nonlinear dynamics of these two studied models of nonlinear dynamics is reflected in the analogy of the sources of nonlinearity in the number of unbalanced pairs of balls on the corresponding diameters of radial ball bearings and with the number of shafts that carry unbalanced discs. The dynamic nonlinearity caused by a pair of two balls of different mass densities, on a single diameter of radial ball bearing rotating on a balanced shaft with discs, corresponds to the dynamic nonlinearity caused by an unbalanced shaft on a rotating shaft supported on a balanced radial ball bearing (see Refs [13-16]).

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